

FROM COMBINATORICS TO LARGE DEVIATIONS FOR THE INVARIANT MEASURES OF SOME MULTICLASS PARTICLE SYSTEMS

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ABSTRACT. We prove large deviation principles (LDP) for the invariant measures of the multiclass totally asymmetric simple exclusion process (TASEP) and the multiclass Hammersely-Aldous-Diaconis (HAD) process on a torus. The proof is based on a combinatorial representation of the measures in terms of a *collapsing procedure* introduced in [2] for the 2-class TASEP and then generalized in [9], [10] and [11] to the multiclass TASEP and the multiclass HAD process. The rate functionals are written in terms of variational problems that we solve in the cases of 2-class processes.

1. INTRODUCTION

In recent years several new and interesting results have been obtained in the study of fluctuations of interacting particle systems. Some of these results concern the computations of large deviations rate functionals for specific models.

Given a stochastic interacting particle system, a problem of interest is the determination of its invariant measures. When the model is not reversible and the detailed balance does not hold, this can be a difficult task. Typical examples are boundary driven stochastic lattice gases. Depending on the model you can have available some representations of the invariant measures or not. We will give in this introduction a short outline of some of the recent progress in computation of the rate functionals of large deviations for the empirical measures of the invariant measures in models of this type. The results are interesting for several reasons. One reason is that the measures have often long range correlations and the corresponding rate functionals are not local. They have a structure very different from the one obtained in the case for example of Gibbs measures where you have an integration of a function of the density profile. Another reason is that interacting particle systems are very effective models of statistical mechanics and the results obtained give insight for the behavior of more complex models or real systems.

From one side there are combinatorial representations for the invariant measures of exclusion like models starting from which it is possible to compute the corresponding rate functionals (see [5] for a recent review and references therein). These combinatorial representations are built up from products of operators satisfying appropriate commutation relations.

A different approach is based on dynamical arguments. Fluctuations of the invariant measure can be recovered from fluctuations of paths of the processes. The static rate functional for the invariant measure is then obtained solving a variational problem for the dynamic rate functional. This leads to an Hamilton-Jacobi equation as a central object (see [3] for a recent review and references therein). Differently from the exact solution approach the dynamic one is insensitive to small perturbations of the dynamics corresponding to the same macroscopic structure of fluctuations. Nevertheless the Hamilton-Jacobi equation is, in general, difficult to solve.

In this paper we prove large deviation principles (LDP) for the invariant measures of the multiclass totally asymmetric simple exclusion process (TASEP) and the multiclass Hammersely-Aldous-Diaconis (HAD) process. Our proof is based on an exact combinatorial representation of the invariant measures. Configurations of particles distributed according to the invariant measures are constructed applying a deterministic transformation, the *collapsing procedure*, to configurations of particles distributed according to product of uniform measures. We generalize the collapsing procedure up to let it act on positive measures. An application of the contraction principle allows one to obtain the final result. The rate functionals that we obtain are not local. In the case of 2 class models they are obtained from a geometric construction on density profiles.

As our results are obtained from the contraction principle, the rate functionals are naturally expressed as infimum of auxiliary functionals. Several non local functionals obtained as rate functionals of particle systems are represented in terms of either infimum or supremum of auxiliary functionals. An interesting question is whether there is always a representation of them as an infimum. More precisely if those rate functionals can be obtained from wider LDP using the contraction principles. This is the case of the present paper. This is also the case of the invariant measures for the TASEP with boundary sources as represented in [7]. The corresponding application of the contraction principle has been done in [3]. In [3] it is also suggested that this could be the case for the KMP model with boundary sources. A rate functional obtained from contraction of a convex rate functional is not necessarily convex, this is also the case of the present result.

The paper is organized as follows. In section 2 we define the multiclass TASEP and the multiclass HAD process constructing them using the basic coupling. In section 3 we describe the *collapsing procedure* as it acts on configurations of particles. We discuss also briefly its main properties. In section 4 we show how the collapsing procedure is used to construct the invariant measures of the processes. In section 5 we generalize the collapsing procedure defining its action on positive measures. We discuss also its main properties. In section 6 we define empirical measures and relate them to the collapsing procedure. In section 7 we derive, from well known results, large deviation principles for uniform distributions. In section 8 we derive LDP for the invariant measures of the 2-class TASEP and the 2-class HAD process. Using the contraction principle we have the rate functionals in a variational form for which we can find the unique minimum in terms of a *concave envelope* construction. These are theorems 8.2 and 8.5 that are the main results of the paper. The rate functionals are not convex. We discuss also in variational terms the typical density of first class particles when the total density is known and the typical total density when the density of first class particles is known. In section 9 we derive LDP for the multiclass TASEP and the multiclass HAD process. The rate functionals are written in terms of variational problems. It is interesting to study such problems and others proposed in the section. We obtain also a recursive relation.

The combinatorial constructions of [2], [9], [10], [11] are different from the original solution of the 2-class TASEP obtained in [6], that is based on products of non commuting operators. It is interesting to derive the same result of the present paper starting from this alternative solution. A probably different representation of the same rate functional will maybe appear. It is also interesting to study the problem using the dynamic approach. The major problem here is the lack of a complete dynamical LDP. A problem of interest is also the study of the variational problem (9.1) for models with more than 2 classes of particles.

To avoid confusion we remark that we use similar symbols for mathematical objects that play a similar role in the TASEP, the HAD process and in the general framework of positive measures. We use also the same symbol \mathbb{C} to indicate the *collapsing operator* both when it acts on configurations of particles and on positive measures. This is due to the fact that the second one is a natural generalization of the previous one.

2. MULTICLASS MODELS

2.1. TASEP. The totally asymmetric simple exclusion process (TASEP) is a model of stochastic jumping particles satisfying an exclusion rule. Let $\mathbb{Z}_N := \mathbb{Z}/(N\mathbb{Z})$ be the discrete one dimensional torus with N sites. Every site $x \in \mathbb{Z}_N$ can be either empty or occupied by a particle. The state space of the process is $X_N := \{0, 1\}^{\mathbb{Z}_N}$. Given $\eta \in X_N$ a configuration of particles, we will say that the site $x \in \mathbb{Z}_N$ is occupied by a particle if $\eta(x) = 1$ and empty otherwise. Every particle waits an exponential time of rate one and then tries to jump to its nearest neighbor site to the left. If the site is already occupied by another particle then the jump is suppressed. This informal description can be summarized by the following generator of the dynamics

$$L_N f(\eta) := \sum_{x \in \mathbb{Z}_N} (f(\eta^{e_x}) - f(\eta))$$

where e_x is the oriented bond $(x, x+1)$ and η^{e_x} is the configuration of particles obtained from η rearranging the values at the extremes of e_x in decreasing order, according to the following definition

$$\eta^{e_x}(z) := \begin{cases} \eta(z) & \text{if } z \neq x, x+1, \\ \max\{\eta(x), \eta(x+1)\} & \text{if } z = x, \\ \min\{\eta(x), \eta(x+1)\} & \text{if } z = x+1. \end{cases}$$

A generalization of the previous model is obtained labeling some of the particles as *first class particles* and the remaining ones as *second class particles*. When a first class particle tries to jump over a second class particle it succeeds and the two particles exchange their positions. When a second class particle tries to jump over a first class particle the jump is suppressed. The natural state space for such a process is $\{0, 1, 2\}^{\mathbb{Z}_N}$ obtained from the choice of assigning value 0 to empty sites, value 1 to sites occupied by a first class particle and value 2 to sites occupied by second class particles. We will instead describe a configuration with a pair (η_1, η_2) with both η_1 and η_2 elements of X_N . The configuration $\eta_1 \in X_N$ is such that $\eta_1(x) = 1$ when in x there is a first class particle and $\eta_1(x) = 0$ otherwise. The configuration $\eta_2 \in X_N$ is such that $\eta_2(x) = 1$ if in x there is either a first or a second class particle and $\eta_2(x) = 0$ otherwise. We endow X_N with the natural partial order \preceq defined from

$$\eta \preceq \xi \Leftrightarrow \eta(x) \leq \xi(x), \quad \forall x \in \mathbb{Z}_N.$$

By definition $\eta_1 \preceq \eta_2$ so that $(\eta_1, \eta_2) \in I_N^{2,\uparrow}$ where

$$I_N^{k,\uparrow} := \{(\eta_1, \dots, \eta_k) : \eta_i \in X_N, \eta_i \preceq \eta_{i+1}\}.$$

The two descriptions of the state space are equivalent and a bijection between $I_N^{2,\uparrow}$ and $\{0, 1, 2\}^{\mathbb{Z}_N}$ is defined from

$$\xi(x) := \inf \{i : \eta_i(x) = 1, i = 0, 1, 2\},$$

where we defined $\eta_0(x) := 1 - \eta_2(x)$.

The generator of the above described 2-class TASEP is

$$L_N f(\eta_1, \eta_2) := \sum_{x \in \mathbb{Z}_N} (f(\eta_1^{e_x}, \eta_2^{e_x}) - f(\eta_1, \eta_2)) .$$

This generator clearly defines also a jointly, order preserving, evolution of two TASEP, usually called basic coupling.

A further natural generalization, called the k -class TASEP is obtained introducing particles of class up to a fixed natural number $k \leq N$. When a particle of class i tries to jump over a particles of class j with $j > i$ then the positions of the two particles are exchanged. When a particle of class i tries to jump over a particle of class j with $j \leq i$ the jump is suppressed. The state space is now $I_N^{k, \uparrow}$ that is in bijection with $\{0, 1, \dots, k\}^{\mathbb{Z}_N}$. The generator of the dynamics is

$$L_N f(\eta_1, \dots, \eta_k) := \sum_{x \in \mathbb{Z}_N} (f(\eta_1^{e_x}, \dots, \eta_k^{e_x}) - f(\eta_1, \dots, \eta_k)) .$$

This generator clearly defines also a jointly, order preserving, evolution of k TASEP, usually called basic coupling.

2.2. HAD process. Let $\Lambda := \mathbb{R}/\mathbb{Z}$ be the one dimensional torus. The Hammersley-Aldous-Diaconis (HAD) process (after [12] and [1]) is a stochastic evolution on finite subsets of Λ . Let

$$\Omega_N := \{\underline{x} := \{x_1, \dots, x_N\} : x_i \in \Lambda; x_i \neq x_j \ i \neq j\}$$

be the collection of all finite subsets of Λ with N points and let $\Omega = \cup_N \Omega_N$. Labels to points are given in such a way that x_{i+1} is the nearest point of \underline{x} to the right of x_i . Given $\underline{x} \in \Omega$ an initial condition, the HAD process preserves the number of points and is defined as follows. Every point x_i waits an exponential time of rate $|(x_i, x_{i+1}]|$ and then jumps to a point uniformly chosen in $(x_i, x_{i+1}]$. This dynamics can be easily summarized from the following generator

$$L f(\underline{x}) = \int_{\Lambda} du \ (f(\underline{x}^u) - f(\underline{x})) ,$$

where the set \underline{x}^u is defined from

$$x_i^u := \begin{cases} x_i & \text{if } u \notin (x_i, x_{i+1}] , \\ u & \text{if } u \in (x_i, x_{i+1}] . \end{cases}$$

This formula holds for any $i = 1, \dots, |\underline{x}|$, with the convention $x_{|\underline{x}|+1} := x_1$. With probability one all the points stay distinct along the evolution.

The multiclass HAD process has not a simple and intuitive behavior. As in the case of the TASEP a natural way to define it is through the basic coupling. The state space is $I_{N_1, \dots, N_k}^{\uparrow}$ defined as

$$I_{N_1, \dots, N_k}^{\uparrow} := \left\{ (\underline{x}^{(1)}, \dots, \underline{x}^{(k)}) : \underline{x}^{(i)} \in \Omega_{N_i}; \underline{x}^{(i)} \subseteq \underline{x}^{(i+1)} \right\} .$$

Here $0 \leq N_1 \leq N_2 \leq \dots \leq N_k$ are k natural numbers. This is a natural set to describe points in Λ with associated an integer class from 1 to k . Points are the elements of $\underline{x}^{(k)}$. The class associated to $x_i^{(k)}$ is $\inf \{j : x_i^{(k)} \in \underline{x}^{(j)}\}$. The multiclass dynamics is defined from the following generator

$$L f(\underline{x}^{(1)}, \dots, \underline{x}^{(k)}) = \int_{\Lambda} du \ (f(\underline{x}^{(1),u}, \dots, \underline{x}^{(k),u}) - f(\underline{x}^{(1)}, \dots, \underline{x}^{(k)}))$$

that describes also a joint, inclusion preserving, evolution of k HAD processes. This joint evolution is usually called basic coupling.

Both for the TASEP and the HAD process a natural way to introduce basic coupling is via a graphical construction where all the coupled processes evolve using the same random marks. In the case of the HAD process for example these marks are points of a rate one Poisson point process on the cylinder $\Lambda \times \mathbb{R}^+$. See [9], [10], [11] for a more detailed description.

3. COLLAPSING PARTICLES

To describe the invariant measures of the multiclass processes previously introduced we have to explain a *collapsing procedure* introduced in [2]. We start describing its action on configurations of the TASEP.

Let us call

$$X_{N,M} := \left\{ \eta \in X_N : \sum_{x \in \mathbb{Z}_N} \eta(x) = M \right\} .$$

Given $0 \leq M_1 \leq M_2 \leq N$ two natural numbers, we define a *collapsing operator*

$$\mathbb{C} : X_{N,M_1} \times X_{N,M_2} \rightarrow I_N^{2,\uparrow}$$

that maps the pair of configurations (η_1, η_2) into the pair of configurations

$$\mathbb{C}(\eta_1, \eta_2) := (C_{\eta_2}[\eta_1], \eta_2) .$$

The collapsed configuration $C_{\eta_2}[\eta_1]$ is obtained from a mass preserving (i.e. number of particle preserving) transformation of the configuration η_1 . The transformation is defined algorithmically as follows. Give any order to the particles of the configuration η_1 . Move the first particle of this configuration that is on a site, say x , such that $\eta_2(x) = 0$, to the first site on the right, say y , such that: $\eta_2(y) = 1$ and $\eta_1(y) = 0$. Update the η_1 configuration according to this movement and iterate the procedure using at every step the same order fixed at the beginning. The final configuration does not depend on the specific order chosen.

We briefly discuss some of the properties of this algorithmic transformation. For more details we refer the reader to [2] where this construction was introduced, and to [9], [10], [11] for an interpretation in terms of queue theory.

By definition we have that $C_{\eta_2}[\eta_1] \preceq \eta_2$. Let $E(y, x)$ be the excess of η_1 particles in $[y, x]$ defined as follows

$$E(y, x) := \sum_{z \in [y, x]} (\eta_1(z) - \eta_2(z)) . \quad (3.1)$$

Lemma 3.1. *There is a positive flux of particles across the bond e_x if and only if there exists $y \in \mathbb{Z}_N$ such that $E(y, x) > 0$.*

Proof. If y is such that $E(y, x) > 0$ then in $[y, x]$ there are more particles of η_1 type than of η_2 . During the collapsing procedure some particles will necessarily flow out of $[y, x]$ and this can happens only through e_x . Conversely let us suppose that for any $y \in \mathbb{Z}_N$ we have $E(y, x) \leq 0$. Let us order the particles of η_1 from right to left starting from x and let z_i be the site corresponding to the i particle. Due to the fact that $E(z_1, x) \leq 0$, particle number 1 will be allocated on a site belonging to $[z_1, x]$. Due to the fact that $E(z_2, x) \leq 0$, particle number 2 will be allocated on a site belonging to $[z_2, x]$ (different from the one of particle number 1) and so on. No particles will flow across e_x . \square

Let $[\cdot]_+$ be the positive part defined as

$$[x]_+ := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2. *The total flux of particles across e_x is*

$$J(x) := \sup_{y \in \mathbb{Z}_N} [E(y, x)]_+.$$

Proof. When $J(x) = 0$ this follows directly from the previous lemma. If $J(x) > 0$ we can argue as follows. Clearly $J(x)$ is a lower bound of the total flux, because for any y the excess of η_1 particles $E(y, x)$, if positive, has necessarily to flow across e_x . Let y^* be the first element of \mathbb{Z}_N to the left of x such that $E(y^*, x) = J(x) > 0$. There is no flux of particles across e_{y^*-1} . This follows from the previous lemma and the fact that for any z it holds $E(z, y^* - 1) \leq 0$. In fact, if $z \in [x + 1, y^* - 1]$ then

$$E(z, x) \leq J(x) = E(y^*, x),$$

that implies $E(z, y^* - 1) = E(z, x) - E(y^*, x) \leq 0$. If instead $z \in [y^*, x]$ we have

$$\begin{aligned} E(z, y^* - 1) + E(y^*, x) &= E(z, x) + E(x + 1, x) \\ &\leq J(x) + M_1 - M_2 = E(y^*, x) + M_1 - M_2, \end{aligned}$$

that implies $E(z, y^* - 1) \leq M_1 - M_2 \leq 0$. We established that all the particles flowing across e_x were originally in $[y^*, x]$. From the characterization of y^* we deduce immediately that $E(y^*, z) > 0$ for any $z \in [y^*, x]$. Let us order the particles of η_2 type contained in $[y^*, x]$ from left to right and let z_i be the site corresponding to the particle number i . Remember that there is an excess $J(x)$ of η_1 particles in this interval. Due to the fact that $E(y^*, z_1) > 0$ a particle of type η_1 will be allocated in z_1 . Due to the fact that $E(y^*, z_2) > 0$ a particle of η_1 type will be allocated in z_2 and so on. At the end all the sites z_i will be occupied by η_1 particles and exactly the excess of particles $J(x)$ will flow through e_x . \square

Lemma 3.3. *For any interval $[a, b]$ it holds*

$$\sum_{x \in [a, b]} C_{\eta_2}[\eta_1](x) = \sum_{x \in [a, b]} \eta_1(x) + J(a - 1) - J(b). \quad (3.2)$$

Proof. This property follows directly from the conservation of mass. Equation (3.2) simply states that the number of η_1 type particles that are at the end of the collapsing procedure in the interval $[a, b]$ is obtained from the number of particles present initially plus the number of particles entered from the left side minus the number of particles exit from the right side. \square

The collapsing procedure is defined in a similar way for configurations of the HAD process. Let $0 \leq N_1 \leq N_2$ be two integer numbers. We define the collapsing operator

$$\mathbb{C} : \Omega_{N_1} \times \Omega_{N_2} \rightarrow I_{N_1, N_2}^\dagger$$

that maps the pair of configurations $(\underline{x}, \underline{y})$ into the pair of configurations

$$\mathbb{C}(\underline{x}, \underline{y}) := (C_{\underline{y}}[\underline{x}], \underline{y}).$$

The collapsed configuration $C_{\underline{y}}[\underline{x}]$ is obtained moving to the right points of \underline{x} . The transformation is defined algorithmically as follows. Give any order to points of \underline{x} . Move the first point of \underline{x} that does not belong to \underline{y} to the nearest point of \underline{y} to the right that does

not belong to \underline{x} . Update the \underline{x} configuration according to the previous transformation and iterate the procedure. The final configuration does not depend on the specific order chosen.

Also in this case it is possible to define an excess $E(u, v)$ of \underline{x} points in the interval $[u, v]$

$$E(u, v) := \left| \{ \underline{x} \cap [u, v] \} \right| - \left| \{ \underline{y} \cap [u, v] \} \right|$$

and consequently a flux of \underline{x} particles at v

$$J(v) := \sup_{u \in \Lambda} [E(u, v)]_+ .$$

Note that J is right continuous i.e. it holds $J(v) = J(v^+) := \lim_{\epsilon \downarrow 0} J(v + \epsilon)$. All the lemmas previously listed, appropriately reformulated, holds also in this case. We do not go into details here because the collapsing procedure will be generalized to a wider framework in section 5. We only write down the analogous of equation (3.2) in this case

$$\left| \{ C_{\underline{y}}[\underline{x}] \cap [u, v] \} \right| = \left| \{ \underline{x} \cap [u, v] \} \right| + J(u^-) - J(v) , \quad (3.3)$$

where $J(u^-)$ is the left limit of J at u , i.e. $J(u^-) := \lim_{\epsilon \downarrow 0} J(u - \epsilon)$.

4. INVARIANT MEASURES

4.1. TASEP. For the TASEP the number of particles $\sum_{x \in \mathbb{Z}_N} \eta(x)$ is a conserved quantity. For any fixed integer $M \leq N$, the TASEP with M particles is an irreducible finite state Markov chain on $X_{N,M}$ and consequently has a unique invariant measure. The process has then a one parameter (i.e. M) family of invariant measures that is easily seen to coincide with the family of uniform distributions of M particles on \mathbb{Z}_N

$$\nu_N^M(\eta) = \begin{cases} \binom{N}{M}^{-1} & \text{if } \eta \in X_{N,M} , \\ 0 & \text{otherwise} . \end{cases}$$

These are all the extremal invariant measures; all the remaining are obtained as convex combinations.

The 2-class TASEP conserves the number of first class particles $\sum_{x \in \mathbb{Z}_N} \eta_1(x)$ and the number of second class particles $\sum_{x \in \mathbb{Z}_N} (\eta_2(x) - \eta_1(x))$. For any fixed pair of non negative integer numbers Δ_1 and Δ_2 such that $M_1 := \Delta_1$ and $M_2 := \Delta_1 + \Delta_2 \leq N$, the TASEP with Δ_1 first class particles and Δ_2 second class particles is an irreducible finite state Markov chain and has an unique invariant measure.

The result in [2] states that this invariant measure is

$$\left(\nu_N^{M_1} \times \nu_N^{M_2} \right) \circ \mathbb{C}^{-1} . \quad (4.1)$$

We use the symbol \times to indicate the product of measures. In general given a measure μ and a measurable map T with the symbol $\mu \circ T^{-1}$ we denote the pull-back measure defined from

$$(\mu \circ T^{-1})(A) := \mu(T^{-1}(A))$$

for any measurable set A .

We remark that (4.1) is a measure on $I_N^{2,\uparrow}$. This two parameters (M_1 and M_2) family of invariant measures constitutes all the extremal invariant measures.

To give a combinatorial representation of the invariant measures of the k -class TASEP we need to extend the collapsing procedure of section 3. This extension to the case of more than two classes of particles is contained in [9] and further discussed in [10], [11].

Let $\Delta_1, \dots, \Delta_k$ be k non negative integer numbers such that $\sum_{i=1}^k \Delta_i \leq N$. Let us call also $M_j := \sum_{i=1}^j \Delta_i$. We define a collapsing operator

$$\mathbb{C}_k : X_{N, M_1} \times X_{N, M_2} \times \dots \times X_{N, M_k} \rightarrow I_N^{k, \uparrow}$$

that associates to the configurations (η_1, \dots, η_k) the configurations

$$(\xi_1, \dots, \xi_k) := \mathbb{C}_k(\eta_1, \dots, \eta_k)$$

defined as follows. The configuration ξ_k coincides with η_k . The configuration ξ_{k-1} coincides with $C_{\eta_k}[\eta_{k-1}]$. The configuration ξ_{k-2} coincides with $C_{\eta_k}[C_{\eta_{k-1}}[\eta_{k-2}]]$. In general the configuration ξ_{k-j} is obtained from the composition of j collapsing procedures

$$\xi_{k-j} := C_{\eta_k} [C_{\eta_{k-1}} [\dots C_{\eta_{k-j+1}} [\eta_{k-j}] \dots]] . \quad (4.2)$$

Obviously according to this definition $\mathbb{C} = \mathbb{C}_2$. The result contained in [9] states that the invariant measure for the TASEP with Δ_i i -class particles ($i = 1, \dots, k$) is

$$\left(\nu_N^{M_1} \times \nu_N^{M_2} \times \dots \times \nu_N^{M_k} \right) \circ \mathbb{C}_k^{-1} .$$

This k -parameter family of invariant measures constitutes all the extremal invariant measures.

4.2. HAD. The HAD process conserves the number of points. In Ω_N the unique invariant measure μ_N is given by the support of the values of N i.i.d random variables uniform in Λ . Equivalently the invariant measure is a uniform Poisson point process in Λ , conditioned to have N points. These are all the extremal invariant measures, all the remaining are obtained as convex combinations.

The 2-class HAD process conserves the number of first class points and the number of second class points. Let Δ_1 and Δ_2 be two non negative integer numbers and let $N_1 := \Delta_1$ and $N_2 := \Delta_1 + \Delta_2$. The 2-class HAD process with Δ_1 points of first class and Δ_2 points of second class has a unique invariant measure that has a combinatorial representation in terms of the collapsing operator. In fact the result in [11] states that this invariant measure is

$$(\mu_{N_1} \times \mu_{N_2}) \circ \mathbb{C}^{-1} .$$

These are all the extremal invariant measures, all the remaining are obtained as convex combinations.

The generalization of this combinatorial construction to the case of more than two classes of points is described also in [11]. We proceed as in the case of the TASEP. Let $\Delta_1, \Delta_2, \dots, \Delta_k$ be non negative integer numbers and call $N_j := \sum_{i=1}^j \Delta_i$. We define a collapsing operator

$$\mathbb{C}_k : \Omega_{N_1} \times \Omega_{N_2} \times \dots \times \Omega_{N_k} \rightarrow I_{N_1, \dots, N_k}^{\uparrow}$$

that associates to the configurations $(\underline{x}^{(1)}, \dots, \underline{x}^{(k)})$ the configurations

$$(\underline{y}^{(1)}, \dots, \underline{y}^{(k)}) := \mathbb{C}_k(\underline{x}^{(1)}, \dots, \underline{x}^{(k)})$$

defined as follows. The configuration $\underline{y}^{(k)}$ coincides with $\underline{x}^{(k)}$. The configuration $\underline{y}^{(k-1)}$ coincides with $C_{\underline{x}^{(k)}}[\underline{x}^{(k-1)}]$. The configuration $\underline{y}^{(k-2)}$ coincides with $C_{\underline{x}^{(k)}}[C_{\underline{x}^{(k-1)}}[\underline{x}^{(k-2)}]]$.

In general the configuration $\underline{y}^{(k-j)}$ is obtained from the composition of j collapsing procedures

$$\underline{y}^{(k-j)} := C_{\underline{x}^{(k)}} \left[C_{\underline{x}^{(k-1)}} \left[\dots C_{\underline{x}^{(k-j+1)}} \left[\underline{x}^{(k-j)} \right] \dots \right] \right] . \quad (4.3)$$

The result in [11] states that the invariant measure for the HAD process with Δ_i i-class particles ($i = 1, \dots, k$) is

$$(\mu_{N_1} \times \mu_{N_2} \times \dots \times \mu_{N_k}) \circ \mathbb{C}_k^{-1} .$$

These are all the extremal invariant measures, all the remaining are obtained as convex combinations.

5. COLLAPSING MEASURES

We start with some definitions. The set of positive measures on Λ will be denoted as \mathcal{M} . With \mathcal{M}^0 we will denote the subset of \mathcal{M} of measures ρ absolutely continuous with respect to Lebesgue measure and with $\mathcal{M}^{0,b}$ the subset of \mathcal{M}^0 containing the elements such that their densities satisfy the condition

$$0 \leq \frac{d\rho}{du} \leq 1 \quad a.e. .$$

With abuse of notation we will indicate with ρ both a generic element of \mathcal{M}^0 and its density. Finally, given $m \in \mathbb{R}^+$, we call $\mathcal{M}_m := \{\rho \in \mathcal{M} : \int_{\Lambda} d\rho = m\}$. Likewise we set $\mathcal{M}_m^0 := \mathcal{M}^0 \cap \mathcal{M}_m$ and $\mathcal{M}_m^{0,b} := \mathcal{M}^{0,b} \cap \mathcal{M}_m$.

We define a partial order \preceq on \mathcal{M} saying that

$$\rho_1 \preceq \rho_2 \Leftrightarrow \int_A d\rho_1 \leq \int_A d\rho_2$$

for any measurable $A \subseteq \Lambda$. We then call

$$I^{k,\uparrow} := \{(\rho_1, \dots, \rho_k) : \rho_i \in \mathcal{M}; \rho_i \preceq \rho_{i+1}\} .$$

We want to generalize to this framework the algorithmic constructions illustrated in section 3. We are dealing no more with configurations of particles, but with positive measures. Given $\rho_1 \in \mathcal{M}_{m_1}$ and $\rho_2 \in \mathcal{M}_{m_2}$ with $m_1 \leq m_2$ we want to define a collapsing operator

$$\mathbb{C} : \mathcal{M}_{m_1} \times \mathcal{M}_{m_2} \rightarrow I^{2,\uparrow}$$

that associates to the pair (ρ_1, ρ_2) the pair

$$\mathbb{C}(\rho_1, \rho_2) := (C_{\rho_2}[\rho_1], \rho_2) ,$$

where the collapsed measure $C_{\rho_2}[\rho_1] \preceq \rho_2$ is obtained *moving mass of ρ_1 to the right*. The natural way to define such a procedure is through a generalization of equations (3.2) and (3.3). We start defining the excess of mass $E(u, v)$ of the measure ρ_1 in the interval $[u, v]$ as

$$E(u, v) := \int_{[u, v]} d\rho_1 - \int_{[u, v]} d\rho_2 .$$

By definition $E(u, v)$ is right continuous in v (i.e. $E(u, v^+) := \lim_{\epsilon \downarrow 0} E(u, v + \epsilon) = E(u, v)$) and left continuous in u (i.e. $E(u^-, v) = E(u, v)$). It satisfies also some simple addition rules

$$\begin{aligned} E(a, c) &= E(a, b^-) + E(b, c) & \forall b \in [a, c] , \\ E(a, c) &= E(a, b) + E(b^+, c) & \forall b \in [a, c] . \end{aligned}$$

In the above formulas we use the convention $E(u, u^-) = E(u^+, u) := 0$. Then we introduce the flux of mass across $v \in \Lambda$ defined as

$$J(v) := \sup_{u \in \Lambda} [E(u, v)]_+ .$$

We give the following definition of the collapsing operator.

Definition 5.1. *The collapsed measure $C_{\rho_2}[\rho_1]$ is defined in such a way that for any interval $(a, b]$ it holds*

$$\int_{(a,b]} dC_{\rho_2}[\rho_1] = \int_{(a,b]} d\rho_1 + J(a) - J(b) . \quad (5.1)$$

Note that the action on intervals of this type completely defines the measure.

Let

$$\mathcal{J} := \{v \in \Lambda : \exists u \in \Lambda \text{ s.t. } E(u, v) > 0\} . \quad (5.2)$$

Given $v \in \mathcal{J}$, let u such that $E(u, v) > 0$. From the right continuity in v we have $E(u, v^+) = E(u, v) > 0$ and consequently there exists an $\epsilon > 0$ such that $v + \delta \in \mathcal{J}$ for any $0 \leq \delta \leq \epsilon$. From this we can deduce that $\mathcal{J} = \cup_i \mathcal{J}_i$. Where \mathcal{J}_i are at most countable many disjoint intervals either of the type $[l_i, r_i)$ or (l_i, r_i) . This fact can be proved with an argument very similar to the one used to characterize open sets on \mathbb{R} (see for example section II.11 of [13]). The condition $m_1 \leq m_2$ implies the fact that the strict inclusion $\mathcal{J} \subset \Lambda$ holds. A sketch of the proof is as follows. Let, by contradiction, assume that $\mathcal{J} = \Lambda$. Then for any $v \in \Lambda$ you can prove there exist an interval \mathcal{U}_v whose right closed boundary is v , it is either open or closed at the left boundary and is such that

$$0 < J(v) = \int_{\mathcal{U}_v} d\rho_1 - \int_{\mathcal{U}_v} d\rho_2 .$$

Moreover any other interval with the same property is contained inside \mathcal{U}_v . Then for any $v_1 \neq v_2$ we have that either $\mathcal{U}_{v_1} \cap \mathcal{U}_{v_2} = \emptyset$ or they are one contained inside the other. We define $\mathcal{W}_v : \cup_{\{w \in \Lambda : v \in \mathcal{U}_w\}} \mathcal{U}_w$. Given $v_1 \neq v_2$ then either $\mathcal{W}_{v_1} = \mathcal{W}_{v_2}$ or they are disjoint. Moreover

$$\int_{\mathcal{W}_v} d\rho_1 - \int_{\mathcal{W}_v} d\rho_2 > 0 \quad \forall v \in \Lambda$$

so that at most countable different \mathcal{W}_v can exist and they form a partition of Λ . We finally have

$$\begin{aligned} 0 &\geq m_1 - m_2 = \int_{\Lambda} d\rho_1 - \int_{\Lambda} d\rho_2 \\ &= \sum_{v_i} \left(\int_{\mathcal{W}_{v_i}} d\rho_1 - \int_{\mathcal{W}_{v_i}} d\rho_2 \right) > 0 , \end{aligned}$$

a contradiction. An alternative route is obtained showing with arguments similar to the ones of the next lemma that there exists an element of Λ in a neighborhood of the left boundary of any \mathcal{U}_v that does not belong to \mathcal{J} .

Lemma 5.2. *Consider $v \in \mathcal{J}_i$. If $\mathcal{J}_i = [l_i, r_i)$ then*

$$J(v) = E(l_i, v) .$$

If instead $\mathcal{J}_i = (l_i, r_i)$ then

$$J(v) = E(l_i^+, v) .$$

Proof. Here and hereafter in some proofs we need to distinguish the two cases when $\mathcal{J}_i = [l_i, r_i)$ or $\mathcal{J}_i = (l_i, r_i)$. We will give the proofs in the case $\mathcal{J}_i = [l_i, r_i)$. The proofs for the other case are analogous. Given $v \in \mathcal{J}_i$ we consider w_n a maximizing sequence for $J(v)$, i.e. a sequence such that

$$\lim_{n \rightarrow \infty} E(w_n, v) = J(v) .$$

If $w_n \in (v, l_i)$ then

$$E(w_n, v) = E(w_n, l_i^-) + E(l_i, v) \leq E(l_i, v) , \quad (5.3)$$

where we used the definition of \mathcal{J} and the fact that there exists an element of \mathcal{J}^c , the complement of \mathcal{J} in Λ , in any neighborhood of l_i . Without loss of generality we can then consider $w_n \in [l_i, v]$. Note also that the simple argument in (5.3) also implies that for any $v \in \mathcal{J}_i$ there exists $u \in [l_i, v]$ such that $E(u, v) > 0$.

We prove now that for any $w \in \mathcal{J}_i$ it holds $E(l_i, w) \geq 0$. Let us introduce

$$\mathcal{N} := \{w \in \mathcal{J}_i : E(l_i, w) < 0\} .$$

Due to the fact that $E(u, v)$ is right continuous in v we have that \mathcal{N} is the union of at most countable many intervals either of the type $[a_j, b_j)$ or of the type (a_j, b_j) . We give the proof in the case there are only a finite number of intervals and a_1 , the nearest boundary element to the right of l_i , is the boundary of an interval of the type (a_1, b_1) . As before, the proof for the remaining cases is analogous. As already showed there exists $u \in [l_i, a_1]$ with $E(u, a_1) > 0$. Then we have

$$0 \geq E(l_i, a_1^+) = E(l_i, a_1) = E(l_i, u^-) + E(u, a_1) \geq E(u, a_1) ,$$

a contradiction. This imply that $\mathcal{N} = \emptyset$. Finally we get for the maximizing sequence $w_n \in [l_i, v]$ that

$$E(l_i, v) = E(l_i, w_n^-) + E(w_n, v) \geq E(w_n, v) .$$

This means that $\tilde{w}_n := l_i$ is a maximizing sequence and this implies the first statement of the lemma. \square

Note that a direct consequence of this lemma is that J is right continuous, an important fact to have that definition 5.1 is well posed.

It is possible to introduce a measure γ defined from

$$\int_{(a,b]} d\gamma := J(b) - J(a) .$$

The measure γ is not a positive measure. In fact we have

$$\int_{\Lambda} d\gamma = \lim_{\epsilon \downarrow 0} \int_{(v+\epsilon, v]} d\gamma = \lim_{\epsilon \downarrow 0} (J(v) - J(v+\epsilon)) = 0 . \quad (5.4)$$

Definition (5.1) then becomes

$$C_{\rho_2}[\rho_1] = \rho_1 - \gamma$$

and using (5.4) we derive the conservation of mass

$$\int_{\Lambda} dC_{\rho_2}[\rho_1] = \int_{\Lambda} d\rho_1 - \int_{\Lambda} d\gamma = \int_{\Lambda} d\rho_1 .$$

We give now a simple representation of the collapsed measure $C_{\rho_2}[\rho_1]$. Similar representations hold also for collapsed configurations of particles in the TASEP and points in the HAD process but their generalization to the case of positive measures is not straightforward.

Lemma 5.3. *The collapsed measure $C_{\rho_2}[\rho_1]$ has the following representation*

$$C_{\rho_2}[\rho_1] = \rho_1 \chi_{\mathcal{J}^c} + \rho_2 \chi_{\mathcal{J}} + \sum_i \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) \delta_{r_i} , \quad (5.5)$$

where with the symbol χ_A we denote the characteristic function of the set A and with δ_v the delta measure in v .

Proof. We show that the weight given to any interval $(a, b]$ from the measure defined by the right hand side of (5.1) coincide with the weight given to the same interval from the measure on the right hand side of (5.5). This implies that the two measures coincide.

We need to verify the following identity

$$\begin{aligned} & \int_{(a,b]} d\rho_1 + J(a) - J(b) \\ &= \int_{(a,b] \cap \mathcal{J}^c} d\rho_1 + \int_{(a,b] \cap \mathcal{J}} d\rho_2 + \sum_{r_i \in (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \int_{(a,b] \cap \mathcal{J}} d\rho_1 - \int_{(a,b] \cap \mathcal{J}} d\rho_2 = - \left(\int_{[l_{i^*}, a]} d\rho_1 - \int_{[l_{i^*}, a]} d\rho_2 \right) \chi_{\mathcal{J}}(a) \\ &+ \left(\int_{[l_{j^*}, b]} d\rho_1 - \int_{[l_{j^*}, b]} d\rho_2 \right) \chi_{\mathcal{J}}(b) + \sum_{r_i \in (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right). \end{aligned} \quad (5.6)$$

In the above formula, when $a \in \mathcal{J}$ we call \mathcal{J}_{i^*} the interval to which it belongs and we assume it is of the type $[l_{i^*}, r_{i^*})$, when $b \in \mathcal{J}$ we call \mathcal{J}_{j^*} the interval to which it belongs and we assume it is of the type $[l_{j^*}, r_{j^*})$. The proof in the remaining cases is similar. Note that it is possible to have $i^* = j^*$.

The sum on the right hand side of (5.6) can be written in the following way.

$$\begin{aligned} & \sum_{r_i \in (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) \\ &= \sum_{\mathcal{J}_i \subseteq (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) + \left(\int_{\mathcal{J}_{i^*}} d\rho_1 - \int_{\mathcal{J}_{i^*}} d\rho_2 \right) \chi_{\mathcal{J}}(a). \end{aligned} \quad (5.7)$$

Using this formula the right hand side of (5.6) becomes

$$\begin{aligned} & \sum_{\mathcal{J}_i \subseteq (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) + \left(\int_{(a, r_{i^*})} d\rho_1 - \int_{(a, r_{i^*})} d\rho_2 \right) \chi_{\mathcal{J}}(a) \\ &+ \left(\int_{[l_{j^*}, b]} d\rho_1 - \int_{[l_{j^*}, b]} d\rho_2 \right) \chi_{\mathcal{J}}(b) = \sum_i \left(\int_{\mathcal{J}_i \cap (a,b]} d\rho_1 - \int_{\mathcal{J}_i \cap (a,b]} d\rho_2 \right), \end{aligned}$$

that clearly coincides with the left hand side of (5.6). \square

We have the following simple properties.

Lemma 5.4. *The measure $C_{\rho_2}[\rho_1]$ is a positive measure.*

Proof. We use the representation (5.5). The first two terms on the right hand side are clearly positive. In the case $\mathcal{J}_i = [l_i, r_i)$ we have that

$$\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 = E(l_i, r_i^-) \geq 0,$$

as shown during the proof of lemma 5.2. This implies that also the third term is positive. An analogous argument holds in the case $\mathcal{J}_i = (l_i, r_i]$. \square

Lemma 5.5. *It holds*

$$C_{\rho_2}[\rho_1] \preceq \rho_2 . \quad (5.8)$$

Proof. Using formula (5.5) we obtain that condition (5.8) is equivalent to

$$\rho_1 \chi_{\mathcal{J}^c} - \rho_2 \chi_{\mathcal{J}^c} + \sum_i \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) \delta_{r_i} \preceq 0 . \quad (5.9)$$

We show that the weight associated to any interval $(a, b]$ from the measure on the left hand side of (5.9) is negative and this implies the statement of the lemma.

We need to prove that for any $(a, b]$ it holds

$$\int_{(a,b] \cap \mathcal{J}^c} d\rho_1 - \int_{(a,b] \cap \mathcal{J}^c} d\rho_2 + \sum_{r_i \in (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) \leq 0 . \quad (5.10)$$

We use the fact that

$$\begin{aligned} & \int_{(a,b] \cap \mathcal{J}^c} d\rho_1 - \int_{(a,b] \cap \mathcal{J}^c} d\rho_2 \\ &= \int_{(a,b]} d\rho_1 - \int_{(a,b]} d\rho_2 - \sum_i \left(\int_{\mathcal{J}_i \cap (a,b]} d\rho_1 - \int_{\mathcal{J}_i \cap (a,b]} d\rho_2 \right) \\ &= \int_{(a,b]} d\rho_1 - \int_{(a,b]} d\rho_2 - \sum_{\mathcal{J}_i \subseteq (a,b]} \left(\int_{\mathcal{J}_i} d\rho_1 - \int_{\mathcal{J}_i} d\rho_2 \right) \\ &\quad - \left(\int_{(a,r_{i^*})} d\rho_1 - \int_{(a,r_{i^*})} d\rho_2 \right) \chi_{\mathcal{J}}(a) - \left(\int_{[l_{j^*}, b]} d\rho_1 - \int_{[l_{j^*}, b]} d\rho_2 \right) \chi_{\mathcal{J}}(b) . \end{aligned} \quad (5.11)$$

In the above formulas we use for the labels i^* and j^* and the corresponding intervals, the same convention as in lemma 5.3. Using (5.11) and equation (5.7) we obtain that the left hand side of (5.10) is equal to

$$\begin{aligned} & \int_{(a,b]} d\rho_1 - \int_{(a,b]} d\rho_2 + \left(\int_{[l_{i^*}, a]} d\rho_1 - \int_{[l_{i^*}, a]} d\rho_2 \right) \chi_{\mathcal{J}}(a) \\ & - \left(\int_{[l_{j^*}, b]} d\rho_1 - \int_{[l_{j^*}, b]} d\rho_2 \right) \chi_{\mathcal{J}}(b) . \end{aligned} \quad (5.12)$$

In the case $i^* \neq j^*$ we have that (5.12) can be written as

$$\begin{aligned} & E(l_{i^*}, l_{j^*}^-) \chi_{\mathcal{J}}(a) \chi_{\mathcal{J}}(b) + E(l_{i^*}, b) \chi_{\mathcal{J}}(a) \chi_{\mathcal{J}^c}(b) \\ & + E(a^+, l_{j^*}^-) \chi_{\mathcal{J}^c}(a) \chi_{\mathcal{J}}(b) + E(a^+, b) \chi_{\mathcal{J}^c}(a) \chi_{\mathcal{J}^c}(b) \end{aligned}$$

and all the terms are non positive.

The case $i^* = j^*$ can happen only when $a \in \mathcal{J}$ and $b \in \mathcal{J}$. In this case we have that (5.12) is equal either to 0 or to

$$\int_{\Lambda} d\rho_1 - \int_{\Lambda} d\rho_2 \leq 0 .$$

□

When both ρ_1 and ρ_2 belong to \mathcal{M}^0 then $E(u, v)$ is continuous in u and v and consequently $E(l_i, r_i^-) = E(l_i, r_i)$. From what we proved in lemma 5.2 we have that $E(l_i, r_i^-) \geq 0$. From the fact that $r_i \in \mathcal{J}^c$ we have that $E(l_i, r_i) \leq 0$. This implies that $E(l_i, r_i) = 0$.

Then the third term on the right hand side of (5.5) is not present and in this case we simply have

$$C_{\rho_2}[\rho_1] = \rho_1 \chi_{\mathcal{J}^c} + \rho_2 \chi_{\mathcal{J}} .$$

This means that also $C_{\rho_2}[\rho_1] \in \mathcal{M}^0$ with a density a.e. given by

$$C_{\rho_2}[\rho_1](v) = \begin{cases} \rho_2(v) & \text{if } v \in \mathcal{J} , \\ \rho_1(v) & \text{if } v \in \mathcal{J}^c . \end{cases} \quad (5.13)$$

The collapsing operator \mathbb{C} is not continuous with respect to the weak topology. This can be easily seen from the following example. Consider the sequences of measures $\rho_1^{(n)} := \delta_{(\frac{1}{2} + \frac{1}{n})}$ and $\rho_2^{(n)} := \delta_{\frac{1}{2}} + \delta_{\frac{3}{4}}$. Clearly we have

$$C_{\rho_2^{(n)}}[\rho_1^{(n)}] = \delta_{\frac{3}{4}} .$$

Moreover it holds

$$\rho_1^{(n)} \xrightarrow{n \rightarrow \infty} \rho_1 \quad \text{and} \quad \rho_2^{(n)} \xrightarrow{n \rightarrow \infty} \rho_2$$

with $\rho_1 := \delta_{\frac{1}{2}}$, $\rho_2 := \delta_{\frac{1}{2}} + \delta_{\frac{3}{4}}$ and convergence is in the weak topology. We have also

$$C_{\rho_2}[\rho_1] = \delta_{\frac{1}{2}} .$$

As a consequence

$$\lim_{n \rightarrow \infty} \mathbb{C}(\rho_1^{(n)}, \rho_2^{(n)}) \neq \mathbb{C}(\rho_1, \rho_2) = \mathbb{C}\left(\lim_{n \rightarrow \infty} (\rho_2^{(n)}, \rho_1^{(n)})\right) .$$

Nevertheless it holds the following continuity result.

Lemma 5.6. *The collapsing operator \mathbb{C} is continuous with respect to the weak topology on $\mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0$.*

Proof. Let $(\rho_1^{(n)}, \rho_2^{(n)}) \in \mathcal{M}_{m_1} \times \mathcal{M}_{m_2}$ be a sequence of measures weakly converging to $(\rho_1, \rho_2) \in \mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0$. Then for any fixed $v \in \Lambda$ we define the following nondecreasing continuous functions on the interval $[0, 1]$

$$G_1(u) := \begin{cases} \frac{\int_{[v-u, v]} d\rho_1}{m_1} & \text{if } u \in [0, 1) , \\ 1 & \text{if } u = 1 , \end{cases}$$

$$G_2(u) := \begin{cases} \frac{\int_{[v-u, v]} d\rho_2}{m_2} & \text{if } u \in [0, 1) , \\ 1 & \text{if } u = 1 . \end{cases}$$

Likewise using the measures $\rho_1^{(n)}$ and $\rho_2^{(n)}$, we define the functions $G_1^{(n)}$ and $G_2^{(n)}$ that are nondecreasing but not necessarily continuous. The weak convergence implies the pointwise convergence of $G_1^{(n)}$ to G_1 and $G_2^{(n)}$ to G_2 . Monotonicity of all the G , and continuity of the limit functions imply also the uniform convergence. Consider for example $G_1^{(n)}$ and G_1 . Fix an arbitrary ϵ , let

$$u_i := \inf \{u : G_1(u) = i\epsilon\} ; \quad i = 0, \dots, \left\lceil \frac{m_1}{\epsilon} \right\rceil$$

and define also $u_{\lceil \frac{m_1}{\epsilon} \rceil + 1} := 1$. For any i let $n(i)$ be such that for any $n > n(i)$ it holds $|G_1^{(n)}(u_i) - G_1(u_i)| \leq \epsilon$. Let also $n^* := \max_i \{n(i)\}$. Monotonicity says that when $u \in [u_i, u_{i+1})$

$$G_1^{(n)}(u_i) \leq G_1^{(n)}(u) \leq G_1^{(n)}(u_{i+1})$$

and

$$G_1(u_i) \leq G_1(u) \leq G_1(u_{i+1}) .$$

These inequalities imply

$$|G_1^{(n)}(u) - G_1(u)| \leq \max \left\{ |G_1^{(n)}(u_i) - G_1(u_{i+1})|, |G_1^{(n)}(u_{i+1}) - G_1(u_i)| \right\} .$$

When $n > n^*$ both terms inside the max are $\leq 2\epsilon$. Let us show this for example for the second one. It holds

$$|G_1^{(n)}(u_{i+1}) - G_1(u_i)| \leq |G_1^{(n)}(u_{i+1}) - G_1(u_{i+1})| + |G_1(u_{i+1}) - G_1(u_i)|$$

and both terms on the right hand side are $\leq \epsilon$. This implies the uniform convergence of $G_1^{(n)}$ to G_1 . The uniform convergence of $G_1^{(n)}$ and $G_2^{(n)}$ implies the uniform convergence of $[G_1^{(n)} - G_2^{(n)}]_+$ to $[G_1 - G_2]_+$. This implies the convergence of

$$J^{(n)}(v) = \sup_{u \in [0,1]} [G_1^{(n)}(u) - G_2^{(n)}(u)]_+ \quad (5.14)$$

to

$$J(v) = \sup_{u \in [0,1]} [G_1(u) - G_2(u)]_+ \quad (5.15)$$

for any $v \in \Lambda$. To prove this simple fact we see that, calling u^* a maximum point in (5.15),

$$\begin{aligned} \liminf_{n \rightarrow \infty} J^{(n)}(v) &= \liminf_{n \rightarrow \infty} \left(\sup_{u \in [0,1]} [G_1^{(n)}(u) - G_2^{(n)}(u)]_+ \right) \\ &\geq \liminf_{n \rightarrow \infty} [G_1^{(n)}(u^*) - G_2^{(n)}(u^*)]_+ = J(v) . \end{aligned}$$

On the opposite direction, let us call w_k^n a maximizing sequence in (5.14), i.e. a sequence such that

$$J^{(n)}(v) = \lim_{k \rightarrow \infty} [G_1^{(n)}(w_k^n) - G_2^{(n)}(w_k^n)]_+ .$$

Consider now a sequence $k(n)$ such that

$$\left| J^{(n)}(v) - [G_1^{(n)}(w_{k(n)}^n) - G_2^{(n)}(w_{k(n)}^n)]_+ \right| < \frac{1}{n} .$$

It holds

$$\limsup_{n \rightarrow \infty} J^{(n)}(v) = \limsup_{n \rightarrow \infty} [G_1^{(n)}(w_{k(n)}^n) - G_2^{(n)}(w_{k(n)}^n)]_+ .$$

We then obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} J^{(n)}(v) &= \limsup_{n \rightarrow \infty} [G_1^{(n)}(w_{k(n)}^n) - G_2^{(n)}(w_{k(n)}^n)]_+ \\ &= \limsup_{n \rightarrow \infty} [G_1(w_{k(n)}^n) - G_2(w_{k(n)}^n)]_+ \leq J(v) . \end{aligned}$$

We conclude the proof showing that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{(a,b]} dC_{\rho_2^{(n)}} [\rho_1^{(n)}] \\ &= \lim_{n \rightarrow \infty} \left(\int_{(a,b]} d\rho_1^{(n)} + J^{(n)}(a) - J^{(n)}(b) \right) \\ &= \int_{(a,b]} d\rho_1 + J(a) - J(b) = \int_{(a,b]} dC_{\rho_2} [\rho_1] . \end{aligned}$$

The convergence of $\int_{(a,b]} d\rho_1^{(n)}$ to $\int_{(a,b]} d\rho_1$ follows from the weak convergence of $\rho_1^{(n)}$ to ρ_1 . The pointwise convergence of $J^{(n)}$ to J has been shown above. This implies the statement of the lemma. \square

Also in this framework the collapsing operators \mathbb{C}_k are defined likewise in (4.2) and (4.3) in the cases of particles configurations. The continuity properties of \mathbb{C}_k are derived also from lemma 5.6.

6. EMPIRICAL MEASURES

Given a configuration $\eta \in X_N$ we associate to it its empirical measure $\pi_N(\eta)$. This is an element of \mathcal{M} defined as

$$\pi_N(\eta) := \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \eta(x) \delta_{\frac{x}{N}} .$$

Given a collection (η_1, \dots, η_k) of k configurations of X_N we will write

$$\pi_N^k(\eta_1, \dots, \eta_k) := (\pi_N(\eta_1), \dots, \pi_N(\eta_k)) .$$

It is easy to check that

$$\pi_N(C_{\eta_2}[\eta_1]) = C_{\pi_N(\eta_2)}[\pi_N(\eta_1)] . \quad (6.1)$$

This is an important identity that justifies the fact that we used the same symbols for the collapsing operators in different frameworks. Note in fact that in (6.1) we are using the same symbol C with different meanings. Equation (6.1) is the key identity to check the following commutation property

$$\pi_N^2 \circ \mathbb{C} = \mathbb{C} \circ \pi_N^2 . \quad (6.2)$$

The validity of (6.2) directly implies its generalization

$$\pi_N^k \circ \mathbb{C}_k = \mathbb{C}_k \circ \pi_N^k . \quad (6.3)$$

Note that the same commutation relations hold also in the case we had given a slightly different definition of empirical measure associated to a configuration of particles of the TASEP. Sometimes the empirical measure associated to a configuration $\eta \in X_N$ is defined as an element of $\mathcal{M}^{0,b}$ whose density is a.e.

$$\pi_N(\eta)(u) = \sum_{x \in \mathbb{Z}_N} \eta(x) \chi_{[\frac{x}{N} - \frac{1}{2N}, \frac{x}{N} + \frac{1}{2N})}(u) .$$

Also in this case (6.1), (6.2) and (6.3) hold.

In the case of the HAD process there is not a natural scale parameter as in the case of the TASEP where it is the size of the lattice. We will consider families, with index a natural number N , of HAD models containing $[Nm]$ particles, with m a positive real number and $[\cdot]$ the integer part. The scale parameter is N and for any configuration $\underline{x} \in \Omega$ we define the empirical measure as

$$\pi_N(\underline{x}) := \frac{1}{N} \sum_i \delta_{x_i} .$$

Note that we do not require in this definition that N coincides with $|\underline{x}|$. As in the case of the TASEP, given a collection $(\underline{x}^{(1)}, \dots, \underline{x}^{(k)})$ of k configurations of Ω we will write

$$\pi_N^k(\underline{x}^{(1)}, \dots, \underline{x}^{(k)}) := \left(\pi_N(\underline{x}^{(1)}), \dots, \pi_N(\underline{x}^{(k)}) \right) .$$

Also in this framework the analogous of (6.1), (6.2) and (6.3) hold.

7. LARGE DEVIATIONS FOR UNIFORM DISTRIBUTIONS

In this section we quickly derive, from well known results, large deviations principles for the invariant measures of the TASEP and the HAD process with particles of only one class.

Given a sequence μ_N of probability measures on a Polish metric space X we say that it satisfies a large deviation principle (LDP) with parameter N and rate function $I : X \rightarrow \mathbb{R}^+ \cup \{+\infty\}$, if

$$\begin{aligned} \limsup_{N \rightarrow +\infty} N^{-1} \log \mu_N(C) &\leq - \inf_{x \in C} I(x) & \forall C \subseteq X, C \text{ closed}; \\ \liminf_{N \rightarrow +\infty} N^{-1} \log \mu_N(O) &\geq - \inf_{x \in O} I(x) & \forall O \subseteq X, O \text{ open}. \end{aligned}$$

The rate function I is lower semicontinuous and is called *good* if it has compact level sets.

Let us recall a classical LDP result for sampling without replacement as stated for example in [4].

Let $\underline{y}^{(N)} = \{y_1^{(N)}, \dots, y_N^{(N)}\}$ be N elements of Λ such that

$$\pi_N(\underline{y}^{(N)}) = \frac{1}{N} \sum_{i=1}^N \delta_{y_i^{(N)}} \xrightarrow{N \rightarrow +\infty} \lambda,$$

where \rightarrow is the weak convergence and $\lambda \in \mathcal{M}$. Let ν_N^M be the uniform measure on M -uples of elements of $\underline{y}^{(N)}$, $M \leq N$:

$$\nu_N^M \left(\left\{ y_{i_1}^{(N)}, \dots, y_{i_M}^{(N)} \right\} \right) = \begin{cases} \binom{N}{M}^{-1} & \text{if } i_j \neq i_k \ \forall j \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

This is the measure obtained from a procedure of M samplings without replacement among the N elements of $\underline{y}^{(N)}$. Consider now the case $M = M(N) = [Nm]$, where $[\cdot]$ is the integer part and $m \in (0, 1]$. Then when N diverges the measures $\nu_N^{M(N)} \circ \pi_N^{-1}$ satisfy a LDP on \mathcal{M} equipped with the weak topology with parameter N and with a good and convex rate function given by

$$\begin{cases} mH\left(\frac{\rho}{m} \middle| \lambda\right) + (1-m)H\left(\frac{\lambda-\rho}{1-m} \middle| \lambda\right) & \text{if } \lambda - \rho \in \mathcal{M}_{1-m}, \\ +\infty & \text{otherwise.} \end{cases} \quad (7.1)$$

Where $H(\cdot|\cdot)$ is the relative entropy.

To get the rate functional for the invariant measure of the TASEP with $[Nm]$ particles we need to consider $y_i^{(N)} = \frac{i}{N}$. With this choice the measure λ coincides with the Lebesgue measure and consequently

$$H(\rho|\lambda) = \begin{cases} \int_{\Lambda} \rho(u) \log \rho(u) du & \text{if } \rho \in \mathcal{M}^0, \\ +\infty & \text{otherwise.} \end{cases}$$

In this case the condition for finiteness in (7.1) is equivalent to $\rho \in \mathcal{M}_m^{0,b}$. Observing that in this case the collection of all the M -uples of $\underline{y}^{(N)}$ is in bijection with $X_{N,M}$ (using the correspondence $\eta(x) = 1$ if and only if $\frac{x}{N}$ belongs to the M -uple), we obtain the following result.

Proposition 7.1. *Let ν_N^M be the invariant measure of the TASEP with M particles. When N diverges the family of measures $\nu_N^{[Nm]} \circ \pi_N^{-1}$ satisfy a LDP on \mathcal{M} equipped with weak topology, with parameter N and with good and convex rate function*

$$S_1(\rho) := \begin{cases} \int_{\Lambda} du \, h_m(\rho(u)) & \text{if } \rho \in \mathcal{M}_m^{0,b}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $h_m(x) = x \log \frac{x}{m} + (1-x) \log \frac{1-x}{1-m}$.

We use the symbol S_1 to remark that it is the rate function for a one class system. The parameter m is understood.

A LDP for the invariant measures of the HAD process with particles of only one class follows directly from Sanov theorem (see [4] section 6.2). Sanov theorem states that if X_i are i.i.d random variables taking values in Λ and having common law λ then the empirical measure

$$\frac{1}{N} \sum_i^N \delta_{X_i}$$

satisfies a LDP on \mathcal{M} equipped with the weak topology with parameter N and with a good convex rate functional given by the relative entropy

$$\begin{cases} H(\rho|\lambda) & \text{if } \rho \in \mathcal{M}_1, \\ +\infty & \text{otherwise.} \end{cases}$$

From this general fact we can easily deduce the following proposition as a special case when λ is the Lebesgue measure.

Proposition 7.2. *Let μ_M be the invariant measure of the HAD process with M points and let m be a positive real number. When N diverges the family of measures $\mu_{[Nm]}$ satisfy a large deviation principle on \mathcal{M} equipped with the weak topology with parameter N and with the good and convex rate functional*

$$S_1(\rho) := \begin{cases} \int_{\Lambda} du \, k_m(\rho(u)) & \text{if } \rho \in \mathcal{M}_m^0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $k_m(x) = x \log \frac{x}{m}$.

For simplicity we use the same symbol S_1 already used for the TASEP, but the rate functions are different.

8. LDP FOR 2-CLASS PROCESSES

Theorem 7.1 is immediately generalized to the case of product measures. Consider $0 < m_1 \leq m_2 \leq 1$ real numbers. The family of measures

$$\left(\nu_N^{[Nm_1]} \times \nu_N^{[Nm_2]} \right) \circ (\pi_N^2)^{-1} \quad (8.1)$$

satisfy a LDP on $\mathcal{M} \times \mathcal{M}$ endowed with weak topology, with parameter N and with good and convex rate function given by

$$\begin{cases} \sum_{i=1}^2 \int_{\Lambda} du \, h_{m_i}(\rho_i(u)) & \text{if } \rho_i \in \mathcal{M}_{m_i}^{0,b}, \\ +\infty & \text{otherwise.} \end{cases} \quad (8.2)$$

We are interested in proving a LDP for the empirical measures of the invariant measures of the 2-class TASEP. This means that we are interested in proving a LDP for the sequence of measures

$$\left[\left(\nu_N^{[Nm_1]} \times \nu_N^{[Nm_2]} \right) \circ (\mathbb{C}_2)^{-1} \right] \circ (\pi_N^2)^{-1}, \quad (8.3)$$

that due to identity (6.2) coincides with the sequence of measures

$$\left[\left(\nu_N^{[Nm_1]} \times \nu_N^{[Nm_2]} \right) \circ (\pi_N^2)^{-1} \right] \circ (\mathbb{C}_2)^{-1} . \quad (8.4)$$

Lemma 5.6 suggests that we can apply the contraction principle.

The contraction principle (see for example [4] section 4.2.1) states that if μ_N is a sequence of measures satisfying a large deviation principle on a Polish metric space X with a good rate functional $I(x)$ and $f : X \rightarrow Y$ is a continuous map from X to another Polish metric space Y , then also the sequence of measures $\mu_N \circ f^{-1}$ satisfy a LDP on Y with good rate functional K given by $K(y) = \inf_{\{x \in f^{-1}(y)\}} I(x)$. This formulation can in fact be extended (see remark c at page 127 of [4]) to the case when the map f is continuous only at the elements $x \in X$ such that $I(x) < +\infty$. This is exactly our setting.

Before stating the theorem that we obtain following this strategy we need some facts and notations. Consider an interval $\mathcal{U} := [u^l, u^r] \subset \Lambda$ and a density $\rho(u)$. Let us introduce the extended function

$$F_\rho^\mathcal{U}(u) := \begin{cases} \int_{u^l}^u \rho(w) dw & \text{if } u \in \mathcal{U} , \\ -\infty & \text{otherwise} . \end{cases}$$

Note that in the case of measures in \mathcal{M}^0 we can use the above notation for integration because there is no difference between open and closed intervals. Let us also consider the extended function $\mathbb{F}_\rho^\mathcal{U}$ on the real line \mathbb{R} defined as

$$\mathbb{F}_\rho^\mathcal{U}(u) := \begin{cases} F_\rho^\mathcal{U}(u^l + u) & \text{if } u \in [0, |[u^l, u^r]|] , \\ -\infty & \text{otherwise} . \end{cases}$$

We call $\widehat{\mathbb{F}}_\rho^\mathcal{U}$ the concave envelope of $\mathbb{F}_\rho^\mathcal{U}(u)$. We then define the following extended function on Λ

$$\widehat{F}_\rho^\mathcal{U}(u) := \widehat{\mathbb{F}}_\rho^\mathcal{U}(|[u^l, u]|)$$

Note that the following properties hold

$$\begin{cases} \widehat{F}_\rho^\mathcal{U}(u) \geq F_\rho^\mathcal{U}(u) , \\ \widehat{F}_\rho^\mathcal{U}(u^l) = F_\rho^\mathcal{U}(u^l) = 0 , \\ \widehat{F}_\rho^\mathcal{U}(u^r) = F_\rho^\mathcal{U}(u^r) = \int_{u^l}^{u^r} \rho(w) dw . \end{cases}$$

Finally we call $\rho^\mathcal{U}$ a density on Λ such that $\int_{u^l}^u \rho^\mathcal{U}(w) dw = \widehat{F}_\rho^\mathcal{U}(u)$ for any $u \in \mathcal{U}$ and $\rho^\mathcal{U}(u) \chi_{\mathcal{U}^c}(u) = 0$ a.e.. It can be shown that $\rho^\mathcal{U}$ is a positive measure and it belongs to $\mathcal{M}^{0,b}$ when $\rho \in \mathcal{M}^{0,b}$.

Given ρ_1 and ρ_2 measurable functions on Λ , the set $\mathcal{U} := \{u \in \Lambda : \rho_1(u) = \rho_2(u)\}$ is a.e. equivalent to the disjoint union of at most countable many closed intervals $\mathcal{U}_i := [u_i^l, u_i^r]$. Indeed it is simple to check that the family of sets with this property is a σ -algebra that includes all the open sets. Hence it must include all the Borel sets.

Lemma 8.1. *Let $(\rho_1, \rho_2) \in I^{2,\uparrow} \cap (\mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0)$. We have that the pair $(\psi_1, \psi_2) \in \mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0$ is such that $\mathbb{C}(\psi_1, \psi_2) = (\rho_1, \rho_2)$ if and only if the following conditions are*

satisfied

$$\begin{cases} \psi_2 = \rho_2 , \\ \psi_1(u)\chi_{\mathcal{U}^c}(u) = \rho_1(u)\chi_{\mathcal{U}^c}(u) , \quad a.e. , \\ F_{\psi_1}^{\mathcal{U}_i}(u_i^r) = F_{\rho_2}^{\mathcal{U}_i}(u_i^r) , \quad \forall i , \\ F_{\psi_1}^{\mathcal{U}_i} \geq F_{\rho_2}^{\mathcal{U}_i} , \quad \forall i . \end{cases} \quad (8.5)$$

Proof. We recall that due to the fact that the measures are absolutely continuous with respect to Lebesgue measure, the above functions F are continuous on the interior part of the intervals where they are different from $-\infty$. First we show that given a pair $(\psi_1, \psi_2) \in \mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0$ that satisfies conditions (8.5) then we have $\mathbb{C}(\psi_1, \psi_2) = (\rho_1, \rho_2)$. Clearly we need only to prove that $C_{\rho_2}[\psi_1] = \rho_1$. Remember that for measures belonging to \mathcal{M}^0 the collapsing procedure acts as in (5.13). Conditions number three and four in (8.5) imply that for any $u \in \mathcal{U}_i$ and for any i we have

$$\int_u^{u_i^r} d\psi_1 - \int_u^{u_i^r} d\rho_2 \leq 0 .$$

Let us consider the following subset of \mathcal{U}

$$\tilde{\mathcal{U}} := \bigcup_i \left\{ u \in \mathcal{U}_i : F_{\psi_1}^{\mathcal{U}_i}(u) > F_{\rho_2}^{\mathcal{U}_i}(u) \right\} .$$

From the fact that equality between the F holds at the boundary of each \mathcal{U}_i and from continuity in the interior part we deduce that this is an open set. We show now that $\tilde{\mathcal{U}}$ coincides with the set \mathcal{J} as defined in equation (5.2) for the pair of measures (ψ_1, ρ_2) . Clearly $\tilde{\mathcal{U}} \subseteq \mathcal{J}$. This follows from the fact that for any $u \in \tilde{\mathcal{U}}$ with $u \in \mathcal{U}_i$ it holds

$$\int_{u_i^l}^u d\psi_1 - \int_{u_i^l}^u d\rho_2 > 0 .$$

To prove that $\tilde{\mathcal{U}} = \mathcal{J}$ we need to show that for any $u \in \tilde{\mathcal{U}}^c$ and for any $v \in \Lambda$ it holds

$$\int_v^u d\psi_1 - \int_v^u d\rho_2 \leq 0 .$$

Note that

$$\int_{[v,u] \cap \mathcal{U}^c} d\psi_1 - \int_{[v,u] \cap \mathcal{U}^c} d\rho_2 \leq 0 ,$$

due to the fact that on \mathcal{U}^c we have a.e. that $\psi_1(w) = \rho_1(w)$ and $\rho_1(w) < \rho_2(w)$. We consider first the case $u \in \mathcal{U}^c$. In this case we have

$$\begin{aligned} \int_v^u d\psi_1 - \int_v^u d\rho_2 &= \int_{[v,u] \cap \mathcal{U}^c} d\psi_1 - \int_{[v,u] \cap \mathcal{U}^c} d\rho_2 \\ &+ \sum_{\mathcal{U}_i \subseteq [v,u]} \left(\int_{\mathcal{U}_i} d\psi_1 - \int_{\mathcal{U}_i} d\rho_2 \right) + \left(\int_v^{u_i^{r*}} d\psi_1 - \int_v^{u_i^{r*}} d\rho_2 \right) \chi_{\mathcal{U}}(v) . \end{aligned} \quad (8.6)$$

When $v \in \mathcal{U}$ we called \mathcal{U}_{i*} the interval to which it belongs. All terms on the right hand side of (8.6) are nonpositive. Consider now the case $u \in \mathcal{U} \cap \tilde{\mathcal{U}}^c$ and call \mathcal{U}_{j*} the interval to which it belongs. In this case we need to modify formula (8.6) multiplying the last term on the right hand side by $\chi_{[v,u]}(u_{j*}^l)$ and adding

$$\left(\int_v^u d\psi_1 - \int_v^u d\rho_2 \right) \chi_{[v,u]^c}(u_{j*}^l) + \left(\int_{u_{j*}^l}^u d\psi_1 - \int_{u_{j*}^l}^u d\rho_2 \right) \chi_{[v,u]}(u_{j*}^l) .$$

Both terms are nonpositive. Note that i^* can coincide with j^* .

Conversely we assume that for $(\psi_1, \psi_2) \in \mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0$ holds $\mathbb{C}(\psi_1, \psi_2) = (\rho_1, \rho_2)$ and show that this implies the validity of conditions (8.5).

The validity of the first condition in (8.5) is obvious due to the fact that the collapsing operator \mathbb{C} preserves the second component.

Using (5.13) and the previous statement we have that a.e. it holds the following equality

$$C_{\rho_2}[\psi_1](u) = \psi_1(u)\chi_{\mathcal{J}^c}(u) + \rho_2(u)\chi_{\mathcal{J}}(u) = \rho_1(u) .$$

We multiply both sides by $\chi_{\mathcal{U}^c}(u)$ and obtain

$$(\psi_1(u) - \rho_1(u))\chi_{\mathcal{J}^c \cap \mathcal{U}^c}(u) + (\rho_2(u) - \rho_1(u))\chi_{\mathcal{J} \cap \mathcal{U}^c}(u) = 0 , \quad a.e. . \quad (8.7)$$

Due to the fact that the two terms can be different from zero on disjoint sets we have that (8.7) is equivalent to the two equations

$$(\psi_1(u) - \rho_1(u))\chi_{\mathcal{J}^c \cap \mathcal{U}^c}(u) = 0 , \quad a.e. , \quad (8.8)$$

$$(\rho_2(u) - \rho_1(u))\chi_{\mathcal{J} \cap \mathcal{U}^c}(u) = 0 , \quad a.e. . \quad (8.9)$$

From the fact that $\rho_1(u) < \rho_2(u)$ a.e. on \mathcal{U}^c and from equation (8.9) we deduce that

$$\int_{\mathcal{J}^c \cap \mathcal{U}^c} d\lambda = \int_{\mathcal{U}^c} d\lambda , \quad (8.10)$$

where λ is the Lebesgue measure. Equation (8.8) imposes that $\psi_1(u) = \rho_1(u)$ a.e. on $\mathcal{J}^c \cap \mathcal{U}^c$ and using (8.10) we obtain the validity of the second condition in (8.5).

In the case of absolutely continuous measures we have that \mathcal{J} is an open set and this implies that u_i^l and u_i^r do not belong to \mathcal{J} for any i . In fact let us suppose for example that $u_i^r \in \mathcal{J}$, then there exists a ball $B_{u_i^r}(\epsilon)$ centered in u_i^r such that $B_{u_i^r}(\epsilon) \subseteq \mathcal{J}$ and $\int_{B_{u_i^r}(\epsilon) \cap \mathcal{U}^c} d\lambda > 0$ (recall that λ denotes Lebesgue measure). This is impossible due to (8.10). From the fact that u_i^l and u_i^r belong to \mathcal{J}^c we deduce that

$$J(u_i^l) = J(u_i^r) = 0 .$$

Using this we obtain

$$\int_{u_i^l}^{u_i^r} d\rho_2 = \int_{u_i^l}^{u_i^r} dC_{\rho_2}[\psi_1] = \int_{u_i^l}^{u_i^r} d\psi_1 , \quad (8.11)$$

that is the third condition in (8.5).

Let us suppose that there exists an $u \in \mathcal{U}_i$ such that $F_{\psi_1}^{\mathcal{U}_i}(u) < F_{\rho_2}^{\mathcal{U}_i}(u)$. Then using also (8.11) we obtain

$$\int_u^{u_i^r} d\psi_1 - \int_u^{u_i^r} d\rho_2 = F_{\rho_2}^{\mathcal{U}_i}(u) - F_{\psi_1}^{\mathcal{U}_i}(u) > 0 .$$

This implies $u_i^r \in \mathcal{J}$ that is impossible. As a consequence we obtain the validity of the fourth condition in (8.5). \square

Now we can state and prove our large deviations result.

Theorem 8.2. *Let $0 < m_1 < m_2 \leq 1$ real numbers. Consider the 2-class TASEP on \mathbb{Z}_N with respectively $[Nm_1]$ first class particles and $[Nm_2]$ total particles. When the pair (η_1, η_2) is distributed according to the invariant measure of the process, we have that*

$\pi_N^2(\eta_1, \eta_2)$ satisfies a LDP with parameter N and with good rate function $S_2(\rho_1, \rho_2)$ defined as follows. It takes the value $+\infty$ when $(\rho_1, \rho_2) \notin I^{2,\uparrow} \cap (\mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b})$. When $(\rho_1, \rho_2) \in I^{2,\uparrow} \cap (\mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b})$ it takes the value

$$S_2(\rho_1, \rho_2) = \int_{\mathcal{U}^c} h_{m_1}(\rho_1(u))du + \sum_i \int_{\mathcal{U}_i} h_{m_1}(\rho_1^{\mathcal{U}_i}(u)) + \int_{\Lambda} h_{m_2}(\rho_2(u))du, \quad (8.12)$$

where \mathcal{U}_i are disjoint closed intervals and $\mathcal{U} = \cup_i \mathcal{U}_i$ is a.e. equivalent to the set

$$\{u \in \Lambda : \rho_1(u) = \rho_2(u)\}.$$

The symbol $\rho_1^{\mathcal{U}_i}$ is defined before the statement of lemma 8.1.

Proof. As outlined before we can apply the contraction principle. The sequence of measures in (8.1) satisfies a LDP with good rate functional given by (8.2) and, as shown in lemma 5.6, the map \mathbb{C} is continuous on every point where (8.2) is different from $+\infty$. We immediately get that the sequence of measures (8.4) satisfies a LDP with a good rate given by

$$S_2(\rho_1, \rho_2) = \inf_{\{(\psi_1, \psi_2) \in \mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b} : \mathbb{C}(\psi_1, \psi_2) = (\rho_1, \rho_2)\}} \left(\int_{\Lambda} h_{m_1}(\psi_1(u))du + \int_{\Lambda} h_{m_2}(\psi_2(u))du \right). \quad (8.13)$$

By convention the infimum over an empty set is defined as $+\infty$ and remember that we are calling ψ_i both the measures and the corresponding densities. From (8.13) we see immediately that S_2 is equal to $+\infty$ on $\left(I^{2,\uparrow} \cap (\mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b})\right)^c$.

The case $m_1 = m_2 = m$ is not covered by the theorem (in fact it could be, choosing appropriately an interval \mathcal{U}). In this case the rate functional is different from $+\infty$ only when $\rho_1 = \rho_2$ and consequently $\mathcal{U} = \Lambda$. The rate functional is then

$$S_2(\rho_1, \rho_2) = \begin{cases} \int_{\Lambda} h_m(\rho(u))du & \text{if } \rho_1 = \rho_2 = \rho \in \mathcal{M}_m^{0,b}, \\ +\infty & \text{otherwise.} \end{cases}$$

When $m_1 < m_2$ we have a strict inclusion $\mathcal{U} \subset \Lambda$.

First we prove existence of a minimizer for the variational problem (8.13), then we prove uniqueness and finally we characterize it.

In the case of the TASEP, existence of a minimizer can be derived directly showing that (8.13) is a minimization problem of a lower semicontinuous functional over a compact set.

We show instead a more involved proof that works also in the case of the HAD process. When $(\rho_1, \rho_2) \in I^{2,\uparrow}$ then $\mathbb{C}(\rho_1, \rho_2) = (\rho_1, \rho_2)$. This means that we can modify the infimum in (8.13) restricting to pairs (ψ_1, ψ_2) that satisfy the further condition

$$\left\{ (\psi_1, \psi_2) \in \mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b} : \sum_{i=1}^2 \int_{\Lambda} h_{m_i}(\psi_i(u))du \leq \sum_{i=1}^2 \int_{\Lambda} h_{m_i}(\rho_i(u))du \right\}. \quad (8.14)$$

Due to the fact that (8.2) is a good rate function we have that (8.14) is a compact set. The constraints set in (8.13) is easily seen to be a closed subset of $\mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b}$ (and this holds also without the condition of bounded densities). In fact consider a sequence $(\psi_1^{(n)}, \psi_2^{(n)})$

belonging to this set and converging to $(\psi_1, \psi_2) \in \mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_1}^{0,b}$. Then by lemma 5.6 we have that

$$\mathbb{C}(\psi_1, \psi_2) = \mathbb{C} \left(\lim_{n \rightarrow \infty} (\psi_1^{(n)}, \psi_2^{(n)}) \right) = \lim_{n \rightarrow \infty} \mathbb{C}(\psi_1^{(n)}, \psi_2^{(n)}) = (\rho_1, \rho_2) ,$$

which means that the set is closed. We obtained an infimum of a lower semicontinuous functional over a compact set and the existence of a minimizer follows.

We prove now uniqueness of the minimizer. The functional to be minimized in (8.13) is strictly convex. This follows directly from the fact that the real functions h_{m_i} are strictly convex. The set on which we are minimizing is also a convex subset of $\mathcal{M}_{m_1}^{0,b} \times \mathcal{M}_{m_2}^{0,b}$. This follows directly from its characterization given in lemma 8.1. If (ψ_1, ψ_2) and (ψ_1^*, ψ_2^*) satisfy conditions (8.5) then clearly also the convex combination $(\psi_{1c}, \psi_{2c}) := c(\psi_1, \psi_2) + (1-c)(\psi_1^*, \psi_2^*)$ satisfies the same conditions and this is clearly true also for the additional condition of bounded density. A classical result in convex analysis [8] guarantees uniqueness of the minimizer of a strictly convex functional over a convex set.

Finally we characterize the unique minimizer.

Using conditions (8.5) we write (8.13) as

$$\begin{aligned} S_2(\rho_1, \rho_2) &= \int_{\Lambda} h_{m_2}(\rho_2(u)) du + \int_{\mathcal{U}^c} h_{m_1}(\rho_1(u)) du \\ &\quad + \sum_i \inf_{\{\phi_i \in A_i\}} \int_{\mathcal{U}_i} h_{m_1}(\phi_i(u)) du , \end{aligned}$$

where

$$A_i := \left\{ \phi \in \mathcal{M}^{0,b} : F_{\phi}^{\mathcal{U}_i} \geq F_{\rho_2}^{\mathcal{U}_i}; F_{\phi}^{\mathcal{U}_i}(u_i^r) = F_{\rho_2}^{\mathcal{U}_i}(u_i^r) \right\} . \quad (8.15)$$

We need then to study the variational problems

$$\inf_{\{\phi_i \in A_i\}} \int_{\mathcal{U}_i} h_{m_1}(\phi_i(u)) du . \quad (8.16)$$

Existence and uniqueness of the minimizers can be shown as before. It remains to characterize them.

From the strict convexity in x of the real function $h_m(x)$ and Jensen inequality we have for any interval $[v_1, v_2] \subseteq \Lambda$ and any density ϕ

$$\frac{1}{|[v_1, v_2]|} \int_{v_1}^{v_2} h_m(\phi(u)) du \geq h_m \left(\frac{1}{|[v_1, v_2]|} \int_{v_1}^{v_2} \phi(u) du \right) . \quad (8.17)$$

Moreover this inequality is strict as soon as the density $\phi(u)$ is not a.e. constant. A geometric interpretation of this inequality is the following. Let ϕ and ϕ' be two densities defined on an interval $\mathcal{V} \subset \Lambda$ and let $[v_1, v_2] \subseteq \mathcal{V}$. Consider the case in which $F_{\phi}^{\mathcal{V}}(u) = F_{\phi'}^{\mathcal{V}}(u)$ for any $u \notin (v_1, v_2)$ and the graph of $F_{\phi'}^{\mathcal{V}}$ when $u \in [v_1, v_2]$ linearly interpolates $(v_1, F_{\phi}^{\mathcal{V}}(v_1))$ and $(v_2, F_{\phi}^{\mathcal{V}}(v_2))$. More precisely

$$F_{\phi'}^{\mathcal{V}}(u) = \frac{F_{\phi}^{\mathcal{V}}(v_2) - F_{\phi}^{\mathcal{V}}(v_1)}{|[v_1, v_2]|} (|[v_1, u]|) + F_{\phi}^{\mathcal{V}}(v_1) ; \quad u \in [v_1, v_2] . \quad (8.18)$$

Note that if $\phi \in \mathcal{M}^{0,b}$ then necessarily also $\phi' \in \mathcal{M}^{0,b}$. Inequality (8.17) then simply says that

$$\int_{\mathcal{V}} h_m(\phi(u)) du \geq \int_{\mathcal{V}} h_m(\phi'(u)) du , \quad (8.19)$$

with the strict inequality holding if $\phi(u)$ and $\phi'(u)$ do not coincide a.e.. In the rest of the proof we consider pairs (ϕ, ϕ') whose corresponding $F_{\phi}^{\mathcal{U}_i}(u)$ and $F_{\phi'}^{\mathcal{U}_i}(u)$ are related as before. We can then apply inequality (8.19).

Consider a $\phi \in A_i$, such that $\mathbb{F}_\phi^{\mathcal{U}_i}$ is not a concave function. Then clearly there exists a ϕ' such that $F_{\phi'}^{\mathcal{U}_i}(u) \geq F_\phi^{\mathcal{U}_i}(u)$ for any u . In particular this implies that also $\phi' \in A_i$. Inequality (8.19) implies that ϕ can not be the unique minimizer. The unique minimizer ϕ_i^* of (8.16) has then necessarily $\mathbb{F}_{\phi_i^*}^{\mathcal{U}_i}$ concave.

Let us now consider $\phi, \psi \in A_i$ such that both $\mathbb{F}_\phi^{\mathcal{U}_i}$ and $\mathbb{F}_\psi^{\mathcal{U}_i}$ are concave and moreover there exists an $u \in \mathcal{U}_i$ such that $\mathbb{F}_\phi^{\mathcal{U}_i}(u) > \mathbb{F}_\psi^{\mathcal{U}_i}(u)$. Then ϕ can not be the unique minimizer. We can in fact construct a ϕ' considering an affine function through $(u, \mathbb{F}_\psi^{\mathcal{U}_i}(u))$, obtained using an element in the superdifferential of $\mathbb{F}_\psi^{\mathcal{U}_i}$ at u . We then have

$$F_{\phi'}^{\mathcal{U}_i}(v) \geq \min \left\{ F_\phi^{\mathcal{U}_i}(v), F_\psi^{\mathcal{U}_i}(v) \right\} \quad , \quad \forall v \in \mathcal{U}_i \quad ,$$

so that $\phi' \in A_i$. Inequality (8.19) then implies that ϕ can not be the unique minimizer. The unique minimizer ϕ_i^* is then necessarily such that $\mathbb{F}_{\phi_i^*}^{\mathcal{U}_i}$ is the smallest among all the concave functions that are above $\mathbb{F}_{\rho_2}^{\mathcal{U}_i}$, that is its *concave envelope* $\widehat{\mathbb{F}}_{\rho_2}^{\mathcal{U}_i}$. This shows that on the interval \mathcal{U}_i we have $\phi_i^*(u) = \rho_2^{\mathcal{U}_i}(u) = \rho_1^{\mathcal{U}_i}(u)$ a.e.. \square

The rate functional S_2 is non negative and is zero if and only if $\psi_1(u) = m_1$ a.e. and $\psi_2(u) = m_2$ a.e. and corresponding $\rho_2(u) = m_2$ a.e. and $\rho_1(u) = C_{m_2}[m_1](u) = m_1$ a.e..

The rate functional S_2 is not convex. This can be shown by the following example. Let us consider densities (ρ_1, ρ_2) and (ρ_1^*, ρ_2^*) defined a.e. as

$$\rho_1(u) = \chi_{[\frac{1}{4}, \frac{1}{2}]}(u) \quad ; \quad \rho_1^*(u) = \frac{1}{2} \chi_{[\frac{1}{2}, 1]}(u) \quad ; \quad \rho_2(u) = \rho_2^*(u) = \chi_{[\frac{1}{4}, 1]}(u) \quad .$$

We consider the convex combination

$$(\rho_{1c}, \rho_{2c}) := c(\rho_1, \rho_2) + (1 - c)(\rho_1^*, \rho_2^*) \quad .$$

We have that

$$S_2(\rho_1, \rho_2) = \int_{\Lambda} h_{\frac{3}{4}}(\rho_2(u)) du + \int_{[0, \frac{1}{2}]} h_{\frac{1}{4}}\left(\rho_1^{[0, \frac{1}{2}]}(u)\right) du + \int_{[\frac{1}{2}, 1]} h_{\frac{1}{4}}(\rho_1(u)) du \quad ,$$

where $\rho_1^{[0, \frac{1}{2}]}(u) = \frac{1}{2} \chi_{[0, \frac{1}{2}]}(u)$ a.e.. We have also

$$S_2(\rho_1^*, \rho_2^*) = \int_{\Lambda} h_{\frac{3}{4}}(\rho_2(u)) du + \int_{\Lambda} h_{\frac{1}{4}}(\rho_1^*(u)) du$$

and for any $c \in (0, 1)$

$$S_2(\rho_{1c}, \rho_{2c}) = \int_{\Lambda} h_{\frac{3}{4}}(\rho_2(u)) du + \int_{\Lambda} h_{\frac{1}{4}}(\rho_{1c}(u)) du \quad .$$

Convexity of S_2 would imply the validity for any $c \in [0, 1]$ of the following inequality

$$cS_2(\rho_1, \rho_2) + (1 - c)S_2(\rho_1^*, \rho_2^*) - S_2(\rho_{1c}, \rho_{2c}) \geq 0 \quad . \quad (8.20)$$

If we take the limit $c \uparrow 1$, on the left hand side of (8.20) we obtain

$$\int_{[0, \frac{1}{2}]} h_{\frac{1}{4}}(\rho_1^{[0, \frac{1}{2}]}(u)) du - \int_{[0, \frac{1}{2}]} h_{\frac{1}{4}}(\rho_1(u)) du \quad ,$$

that is clearly strictly negative. This shows that S_2 is not convex.

To simplify notations in the following lemmas the fact that all the measures are absolutely continuous, have bounded densities and have a fixed total mass will be understood. It is understood also the fact that $(\rho_1, \rho_2) \in I^{2,\uparrow}$.

We can still obtain interesting results from the contraction principle. The following identity has to be satisfied

$$\inf_{\rho_1} S_2(\rho_1, \rho_2) = S_1(\rho_2) = \int_{\Lambda} h_{m_2}(\rho_2(u)) du . \quad (8.21)$$

From the microscopic point of view this identity simply derives from the fact that if we forget the labels first and second class and we just look at positions of particles the dynamics that we observe is a TASEP. From the variational point of view we have the following lemma.

Lemma 8.3. *The unique minimizer ρ_1^* in (8.21) such that*

$$S_2(\rho_1^*, \rho_2) = S_1(\rho_2)$$

is given by

$$\rho_1^* = C_{\rho_2}[m_1] , \quad (8.22)$$

where m_1 denotes the measure with a constant density equal to m_1 .

Proof. We have

$$\begin{aligned} \inf_{\{\rho_1\}} \inf_{\{(\psi_1, \psi_2): \mathbb{C}(\psi_1, \psi_2) = (\rho_1, \rho_2)\}} & \left(\int_{\Lambda} h_{m_1}(\psi_1(u)) du + \int_{\Lambda} h_{m_2}(\psi_2(u)) du \right) \\ &= \int_{\Lambda} h_{m_2}(\rho_2(u)) du + \inf_{\psi_1} \int_{\Lambda} h_{m_1}(\psi_1(u)) du \\ &= \int_{\Lambda} h_{m_2}(\rho_2(u)) du . \end{aligned}$$

The unique minimizer in the expression above has a density $\psi_1^*(u) = m_1$ a.e. and correspondingly the unique minimizer ρ_1^* in (8.21) is obtained from the collapsing procedure applied to ψ_1^* , that is (8.22). \square

The minimizer ρ_1^* in (8.22) corresponds to the typical density of first class particles when the system is conditioned to have a total density ρ_2 .

Still from the contraction principle we have that the following identity has to be satisfied

$$\inf_{\rho_2} S_2(\rho_1, \rho_2) = S_1(\rho_1) = \int_{\Lambda} h_{m_1}(\rho_1(u)) du . \quad (8.23)$$

From the microscopic point of view this identity simply derives from the fact that if we forget second class particles and observe only first class particles what we see is a TASEP. Identity (8.23) can be deduced also from purely variational arguments.

Fix the density ρ_1 and call $\mathcal{V} = \cup_i \mathcal{V}_i$ a subset of Λ a.e. equivalent to the subset $\{u \in \Lambda : \rho_1(u) > m_2\}$. The sets \mathcal{V}_i are disjoint closed intervals, $\mathcal{V}_i := [v_i^l, v_i^r]$. To any such an interval we associate an element $w_i^l \in \Lambda$. This element is the nearest $w \in \Lambda$ to the left of v_i^l such that

$$\int_w^{v_i^r} (m_2 - \rho_1(z)) dz = 0 . \quad (8.24)$$

It exists due to the fact that $\int_{\Lambda} (m_2 - \rho_1(z)) dz = m_2 - m_1 > 0$, the function of w given by $\int_w^{v_i^r} (m_2 - \rho_1(z)) dz$ is continuous and condition (8.24) identify a closed set. Consider now the intervals $\tilde{\mathcal{V}}_i := [w_i^l, v_i^r]$.

Lemma 8.4. *The unique minimizer ρ_2^* in (8.23) such that*

$$S_2(\rho_1, \rho_2^*) = S_1(\rho_1) ,$$

has a density a.e. equal to

$$\rho_2^*(u) = \begin{cases} m_2 & \text{if } u \notin \cup_i \tilde{\mathcal{V}}_i , \\ \rho_1(u) & \text{if } u \in \cup_i \tilde{\mathcal{V}}_i . \end{cases} \quad (8.25)$$

Proof. We have

$$\begin{aligned} S_2(\rho_1, \rho_2) - S_1(\rho_1) &= \int_{\Lambda} h_{m_2}(\rho_2(u)) du + \int_{\mathcal{U}^c} h_{m_1}(\rho_1(u)) du \\ &\quad + \sum_i \int_{\mathcal{U}_i} h_{m_1}(\rho_1^{\mathcal{U}_i}(u)) - \int_{\Lambda} h_{m_1}(\rho_1(u)) \\ &= \int_{\mathcal{U}^c} h_{m_2}(\rho_2(u)) du + \sum_i \int_{\mathcal{U}_i} h_{m_2}(\rho_1(u)) \\ &\quad + \sum_i \int_{\mathcal{U}_i} h_{m_1}(\rho_1^{\mathcal{U}_i}(u)) - \sum_i \int_{\mathcal{U}_i} h_{m_1}(\rho_1(u)) . \end{aligned}$$

Where we used the fact that $\rho_1(u)\chi_{\mathcal{U}}(u) = \rho_2(u)\chi_{\mathcal{U}}(u)$ a.e.. We add and subtract $\sum_i \int_{\mathcal{U}_i} h_{m_2}(\rho_2^{\mathcal{U}_i}(u)) du$ and finally we obtain

$$S_2(\rho_1, \rho_2) - S_1(\rho_1) = \int_{\mathcal{U}^c} h_{m_2}(\rho_2(u)) du + \sum_i \int_{\mathcal{U}_i} h_{m_2}(\rho_2^{\mathcal{U}_i}(u)) \geq 0 . \quad (8.26)$$

We used the fact that given an interval \mathcal{V} and a density ρ

$$\int_{\mathcal{V}} (h_{m_2}(\rho(u)) - h_{m_1}(\rho(u))) du = \left(\int_{\mathcal{V}} \rho(u) du \right) \log \frac{m_1(1-m_2)}{m_2(1-m_1)} + \log \frac{1-m_1}{1-m_2} ,$$

depends only on m_1, m_2 and the total mass $\int_{\mathcal{V}} \rho(u) du$. This imply that

$$\int_{\mathcal{U}_i} (h_{m_2}(\rho_1(u)) - h_{m_1}(\rho_1(u))) du = \int_{\mathcal{U}_i} (h_{m_2}(\rho_1^{\mathcal{U}_i}(u)) - h_{m_1}(\rho_1^{\mathcal{U}_i}(u))) du ,$$

due to the fact that $\int_{\mathcal{U}_i} \rho_1(u) du = \int_{\mathcal{U}_i} \rho_1^{\mathcal{U}_i}(u) du$. The right hand side of (8.26) can be zero if and only if $\rho_2(u)\chi_{\mathcal{U}^c}(u) = m_2\chi_{\mathcal{U}^c}(u)$ a.e. and $\rho_2^{\mathcal{U}_i}(u)\chi_{\mathcal{U}_i}(u) = m_2\chi_{\mathcal{U}_i}(u)$ a.e. for any i . This happens if and only if ρ_2 is constructed as in (8.25). Let us show this fact.

Given two intervals $\tilde{\mathcal{V}}_i$ and $\tilde{\mathcal{V}}_j$ then they are either disjoint or one contained inside the other. This follows from the following statement: if $w_i^l \notin [v_j^r, v_i^l]$ then $w_i^l \in (v_i^r, w_j^l)$. To prove the statement observe that, by definition of w_i^l , for any $u \in (w_i^l, v_i^r)$ we have

$$\int_{w_i^l}^u (m_2 - \rho_1(z)) dz + \int_u^{v_i^r} (m_2 - \rho_1(z)) dz = 0$$

and moreover the second integral is strictly negative. As a consequence for any $u \in (w_i^l, v_i^r)$ it holds

$$\int_{w_i^l}^u (m_2 - \rho_1(z)) dz > 0 . \quad (8.27)$$

If $w_i^l \notin [v_j^r, v_i^l]$ then $\int_{w_i^l}^{v_j^r} (m_2 - \rho_1(z)) dz > 0$. Clearly we have also $\int_u^{v_j^r} (m_2 - \rho_1(z)) dz < 0$ for any $u \in (w_i^l, v_i^r)$. As a consequence we deduce that there exists an $u \in (w_i^l, v_i^r)$ such that $\int_u^{v_j^r} (m_2 - \rho_1(z)) dz = 0$ and this implies the above statement.

We consider the subfamily of intervals $\tilde{\mathcal{V}}_{i_k}$ composed by the intervals $\tilde{\mathcal{V}}_i$ that are not contained inside intervals $\tilde{\mathcal{V}}_j$ with $j \neq i$. Note that $\cup_k \tilde{\mathcal{V}}_{i_k} = \cup_i \tilde{\mathcal{V}}_i$.

The fact that (8.25) is a minimizer of (8.26) and consequently also of (8.23) follows from the fact that inequality (8.27) is equivalent to inequality $F_{m_2}^{\tilde{V}_{i_k}} \geq F_{\rho_1}^{\tilde{V}_{i_k}}$ with the equality sign holding only at the boundary of \tilde{V}_{i_k} . This implies the fact that $\hat{F}_{\rho_1}^{\tilde{V}_{i_k}}$ coincides with $F_{m_2}^{\tilde{V}_{i_k}}$ and consequently $\rho_1^{\tilde{V}_{i_k}}(u)\chi_{\tilde{V}_{i_k}}(u) = m_2\chi_{\tilde{V}_{i_k}}(u)$ a.e.. The fact that (8.25) is the unique minimizer can be shown from the fact that (8.26) is strictly positive for different density profiles. \square

The density profile ρ_2^* in (8.25) is the typical total density profile when the system is conditioned to have a density profile ρ_1 of first class particles.

Following the steps of all the proofs presented for the TASEP you see that the only properties of the model that we used are: the strict convexity in x of $h_m(x)$ and the fact that $h_m(x) \geq 0$ with equality if and only if $x = m$. Both properties hold also for $k_m(x)$. As a consequence all the above statements hold also for the HAD process. Starting from theorem 7.2 and proceeding as before we obtain the following result.

Theorem 8.5. *Let $0 < m_1 < m_2$ positive real numbers. Consider the 2-class HAD process on Λ having respectively $[Nm_1]$ first class particles and $[Nm_2]$ total particles. When the pair $(\underline{x}^{(1)}, \underline{x}^{(2)})$ is distributed according to the invariant measure of the process, we have that $\pi_N^2(\underline{x}^{(1)}, \underline{x}^{(2)})$ satisfies a large deviation principle with parameter N and with good rate function $S_2(\rho_1, \rho_2)$ defined as follows. It takes the value $+\infty$ when $(\rho_1, \rho_2) \notin I^{2,\uparrow} \cap (\mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0)$. When $(\rho_1, \rho_2) \in I^{2,\uparrow} \cap (\mathcal{M}_{m_1}^0 \times \mathcal{M}_{m_2}^0)$ it takes the value*

$$S_2(\rho_1, \rho_2) = \int_{\mathcal{U}^c} k_{m_1}(\rho_1(u))du + \sum_i \int_{\mathcal{U}_i} k_{m_1}(\rho_1^{\mathcal{U}_i}(u)) + \int_{\Lambda} k_{m_2}(\rho_2(u))du, \quad (8.28)$$

where \mathcal{U}_i are disjoint closed intervals and $\mathcal{U} = \cup_i \mathcal{U}_i$ is a.e. equivalent to the set

$$\{u \in \Lambda : \rho_1(u) = \rho_2(u)\}.$$

The symbol $\rho_1^{\mathcal{U}_i}$ is defined before the statement of lemma 8.1.

The rate functional S_2 is non negative and is zero if and only if $\psi_1(u) = m_1$ a.e. and $\psi_2(u) = m_2$ a.e. and corresponding $\rho_2(u) = m_2$ a.e. and $\rho_1(u) = C_{m_2}[m_1](u) = m_1$ a.e..

Also in this case the rate functional S_2 is not convex.

The typical density ρ_1^* of first class particles when you condition the system to have a total density ρ_2 is given by (8.22).

The typical total density profile ρ_2^* for the system conditioned to have a density profile ρ_1 of first class particles is given by (8.25).

9. LDP FOR MULTICLASS PROCESSES

Theorem 7.1 is also immediately generalized to the case of product of k measures. Consider $0 < m_1 \leq m_2 \leq \dots \leq m_k \leq 1$ real numbers. The family of measures

$$\left(\nu_N^{[Nm_1]} \times \dots \times \nu_N^{[Nm_k]} \right) \circ \left(\pi_N^k \right)^{-1}$$

satisfies a LDP with parameter N and with good rate functional given by

$$\begin{cases} \sum_{i=1}^k \int_{\Lambda} du h_{m_i}(\rho_i(u)) & \text{if } \rho_i \in \mathcal{M}_{m_i}^{0,b}, \\ +\infty & \text{otherwise.} \end{cases}$$

We are interested in proving a LDP for the empirical measures of the invariant measures of the k -class TASEP. This means that we are interested in proving a LDP for the sequence of measures

$$\left[\left(\nu_N^{[Nm_1]} \times \cdots \times \nu_N^{[Nm_k]} \right) \circ (\mathbb{C}_k)^{-1} \right] \circ \left(\pi_N^k \right)^{-1} ,$$

that due to identity (6.2) coincides with the sequence of measures

$$\left[\left(\nu_N^{[Nm_1]} \times \cdots \times \nu_N^{[Nm_k]} \right) \circ \left(\pi_N^k \right)^{-1} \right] \circ (\mathbb{C}_k)^{-1} .$$

As in the previous section we can apply the contraction principle obtaining the following result.

Theorem 9.1. *Let $0 < m_1 \leq \dots \leq m_k \leq 1$ be real numbers. Consider the k -class TASEP on \mathbb{Z}_N with $[Nm_i]$ particles of class i . When (η_1, \dots, η_k) is distributed according to the invariant measure of the process, we have that $\pi_N^k(\eta_1, \dots, \eta_k)$ satisfies a LDP with parameter N and good rate function $S_k(\rho_1, \dots, \rho_k)$ given by*

$$S_k(\rho_1, \dots, \rho_k) = \inf_{\{(\psi_1, \dots, \psi_k) : \psi_i \in \mathcal{M}_{m_i}^{0,b}, \mathbb{C}_k(\psi_1, \dots, \psi_k) = (\rho_1, \dots, \rho_k)\}} \left(\sum_{i=1}^k \int_{\Lambda} h_{m_i}(\psi_i(u)) du \right) , \quad (9.1)$$

with the convention that the infimum over an empty set is $+\infty$.

Remember that in (9.1) we are indicating with ψ_i both the measure and the corresponding density. The functional S_k is nonnegative and zero if and only if $\rho_i(u) = m_i$ a.e., moreover it takes the value $+\infty$ on $\left[I^{k,\uparrow} \cap \left(\mathcal{M}_{m_1}^{0,b} \times \cdots \times \mathcal{M}_{m_k}^{0,b} \right) \right]^c$.

We obtained an interesting variational problem that will not be studied in this paper. We remark only the following fact. Existence of a minimizer for (9.1) can be proved using the same strategy as in theorem 8.2. Uniqueness of the minimizer is not guaranteed as in the case of 2-class models. We have in fact that when $k > 2$ the set

$$\left\{ (\psi_1, \dots, \psi_k) : \psi_i \in \mathcal{M}_{m_i}^{0,b}, \mathbb{C}_k(\psi_1, \dots, \psi_k) = (\rho_1, \dots, \rho_k) \right\} \quad (9.2)$$

is not necessarily a convex set. This follows from the following example. Let ϵ be any positive real number $< \frac{1}{8}$ and consider the measures defined from the following densities

$$\begin{cases} \psi_1 = 2\chi_{[\frac{1}{8}, \frac{1}{8}+\epsilon]} , \\ \psi_2 = 4\chi_{[0, \epsilon]} + 4\chi_{[\frac{7}{8}, \frac{7}{8}+\epsilon]} , \\ \psi_3 = 4\chi_{[\frac{1}{4}, \frac{1}{4}+\epsilon]} + 8\chi_{[\frac{1}{2}, \frac{1}{2}+\epsilon]} , \end{cases}$$

and

$$\begin{cases} \tilde{\psi}_1 = 2\chi_{[\frac{5}{8}, \frac{5}{8}+\epsilon]} , \\ \tilde{\psi}_2 = 4\chi_{[\frac{3}{8}, \frac{3}{8}+\epsilon]} + 4\chi_{[\frac{3}{4}, \frac{3}{4}+\epsilon]} , \\ \tilde{\psi}_3 = 4\chi_{[\frac{1}{4}, \frac{1}{4}+\epsilon]} + 8\chi_{[\frac{1}{2}, \frac{1}{2}+\epsilon]} . \end{cases}$$

We have that

$$\mathbb{C}_3(\psi_1, \psi_2, \psi_3) = \mathbb{C}_3(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3) = (\rho_1, \rho_2, \rho_3) ,$$

where the measures ρ_i have densities a.e. equal to

$$\begin{cases} \rho_1 = 4\chi_{[\frac{1}{4}, \frac{1}{4}+\frac{\epsilon}{2}]} , \\ \rho_2 = 4\chi_{[\frac{1}{4}, \frac{1}{4}+\epsilon]} + 8\chi_{[\frac{1}{2}, \frac{1}{2}+\frac{\epsilon}{2}]} , \\ \rho_3 = 4\chi_{[\frac{1}{4}, \frac{1}{4}+\epsilon]} + 8\chi_{[\frac{1}{2}, \frac{1}{2}+\epsilon]} . \end{cases}$$

It is easy to check that for the convex combination

$$(\phi_1, \phi_2, \phi_3) = \frac{1}{2}(\psi_1, \psi_2, \psi_3) + \frac{1}{2}(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)$$

we have $\mathbb{C}_3(\phi_1, \phi_2, \phi_3) \neq (\rho_1, \rho_2, \rho_3)$.

The same kind of result is easily derived also for the HAD process starting from theorem 7.2.

Theorem 9.2. *Let $0 < m_1 \leq \dots \leq m_l$ be real numbers. Consider the l -class HAD process on Λ with $[Nm_i]$ points of class $\leq i$. When $(\underline{x}^{(1)}, \dots, \underline{x}^{(l)})$ is distributed according to the invariant measure of the process, we have that $\pi_N^l(\underline{x}^{(1)}, \dots, \underline{x}^{(l)})$ satisfies a LDP with parameter N and with good rate function $S_l(\rho_1, \dots, \rho_l)$ given by*

$$S_l(\rho_1, \dots, \rho_l) = \inf_{\{(\psi_1, \dots, \psi_l): \psi_i \in \mathcal{M}_{m_i}^0, \mathbb{C}_l(\psi_1, \dots, \psi_l) = (\rho_1, \dots, \rho_l)\}} \left(\sum_{i=1}^l \int_{\Lambda} k_{m_i}(\psi_i(u)) du \right) \quad (9.3)$$

with the convention that the infimum over an empty set is $+\infty$.

The functional S_l is nonnegative and zero if and only if $\rho_i(u) = m_i$ a.e.. Moreover it takes the value $+\infty$ on $[I^{l,\uparrow} \cap (\mathcal{M}_{m_1}^0 \times \dots \times \mathcal{M}_{m_l}^0)]^c$.

Still using the contraction principle we obtain the following identity, valid both for the TASEP and the HAD process, whose study from the variational point of view seems to be interesting

$$\inf_{\{\rho_j\}} S_k(\rho_1, \dots, \rho_k) = S_{k-1}(\rho_1, \dots, \hat{\rho}_j, \dots, \rho_k) . \quad (9.4)$$

With the symbol $\hat{\rho}_j$ we indicate the fact that the measure ρ_j is missing. From the microscopic point of view identity (9.4) derives from the fact that if you change the class of j -class particles to $j+1$ the dynamics that you observe is the one of a $(k-1)$ -class process.

We derive now a recursive relation. We derive it for the TASEP but it holds also for the HAD process. To simplify notations the fact that all the measures ρ_i , ψ_i and ϕ_i involved are absolutely continuous, have bounded densities and have fixed total mass will be understood. Also the fact that $(\rho_1, \dots, \rho_k) \in I^{k,\uparrow}$ is understood. We can write (9.1) as

$$S_k(\rho_1, \dots, \rho_k) = \int_{\Lambda} h_{m_k}(\rho_k(u)) du + \inf_{\{(\psi_1, \dots, \psi_{k-1}): \mathbb{C}_k(\psi_1, \dots, \psi_{k-1}, \rho_k) = (\rho_1, \dots, \rho_k)\}} \left(\sum_{i=1}^{k-1} \int_{\Lambda} h_{m_i}(\psi_i(u)) du \right) , \quad (9.5)$$

because if $C_k(\psi_1, \dots, \psi_k) = (\rho_1, \dots, \rho_k)$ then $\psi_k = \rho_k$. Let us call $(\phi_1, \dots, \phi_{k-1}) := \mathbb{C}_{k-1}(\psi_1, \dots, \psi_{k-1})$. We have then that $(\phi_1, \dots, \phi_{k-1}) \in A_{\rho_k}^{k-1}$, where

$$A_{\rho_k}^{k-1} := \left\{ (\phi_1, \dots, \phi_{k-1}) \in I^{k-1,\uparrow} : C_{\rho_k}[\phi_i] = \rho_i, i = 1, \dots, k-1 \right\} .$$

Then equation (9.5) becomes

$$S_k(\rho_1, \dots, \rho_k) = \int_{\Lambda} h_{m_k}(\rho_k(u)) du + \inf_{\{(\phi_1, \dots, \phi_{k-1}) \in A_{\rho_k}^{k-1}\}} \inf_{\{(\psi_1, \dots, \psi_{k-1}): \mathbb{C}_{k-1}(\psi_1, \dots, \psi_{k-1}) = (\phi_1, \dots, \phi_{k-1})\}} \left(\sum_{i=1}^{k-1} \int_{\Lambda} h_{m_i}(\psi_i(u)) du \right)$$

that finally becomes the following recursive relation

$$S_k(\rho_1, \dots, \rho_k) = \int_{\Lambda} h_{m_k}(\rho_k(u)) du + \inf_{\{(\phi_1, \dots, \phi_{k-1}) \in A_{\rho_k}^{k-1}\}} S_{k-1}(\phi_1, \dots, \phi_{k-1}) .$$

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